Exploring $\mathcal{F}(1,2)$-Invariant Graphs

Brian Alspach
University of Newcastle

Joint work with
Ted Dobson, Afsaneh Khodadadpour
Don Kreher and Primoz Šparl

Background

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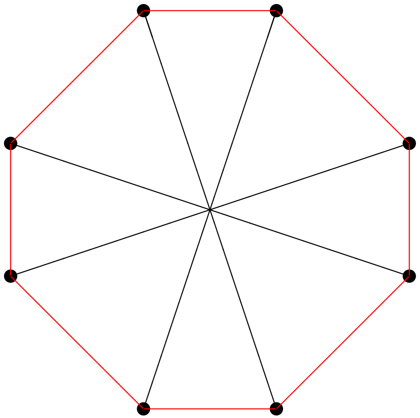
Here are some examples.

Some Examples



Each red edge belongs to a single 4-cycle, whereas, each diameter edge belongs to two 4-cycles. Thus, every automorphism preserves the partition into red edges and black edges. The graph is vertex-transitive as it is a circulant graph.

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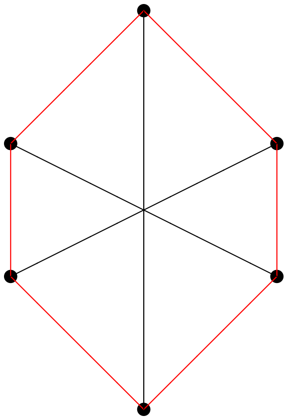

This graph is NOT $\mathcal{F}(1,2)$-invariant because it is isomorphic to $K_{3,3}$ and it is easy to see that no matter how you partition the edge set into a Hamilton cycle and a perfect matching, there is an automorphism not preserving the partition. This also is a circulant graph.

One Cycle

If X is a trivalent vertex-transitive graph of order n possessing a Hamilton cycle C such that $\mathrm{Aut}(X)$ preserves C, then it is clear that $\mathrm{Aut}(X)$ must be a transitive subgroup of D_n, the dihedral group of degree n and order $2n$.

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If $\rho \in \mathrm{Aut}(X)$, then X is a circulant graph.

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Circulant graphs are most easily described as graphs whose vertices are labelled by $\mathbb{Z}_n$ (the ring of integers modulo n), and whose edges are determined by a subset S of $\mathbb{Z}_n$ satisfying $0 \notin S$ and S is inverse-closed (additive inverse). We then join i and j by an edge if and only if $j - i \in S$. S is called the connection set.

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Because we want valency 3, we need an element in the connection set which is an involution. The only possibility is $n/2$ and n even.

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The circulant graph of order 4 with connection set $\{1, 2, 3\}$ is K_4 which is not $\mathcal{F}(1, 2)$-invariant. The circulant graph of order 6 with connection set $\{1, 3, 5\}$ is $K_{3,3}$ which also is not $\mathcal{F}(1, 2)$-invariant.

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Theorem. A circulant graph is $\mathcal{F}(1, 2)$-invariant, where the 2-factor is a Hamilton cycle, if and only if it has even order $n \geq 8$ and it has connection set $\{s, n - s, n/2\}$, where $\gcd(n, s) = 1$.

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It is easy to verify that circulants with connection sets $\{s, n-s, n/2\}$ and $\{1, n/2, n-1\}$ are isomorphic when $\gcd(n,s)=1$. So to within isomorphism there is a unique $\mathcal{F}(1,2)$-invariant circulant graph of even order $n \geq 8$.

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So we can see that the chord incident with u_0 must have its other end a vertex of the form u_ℓ, ℓ odd.

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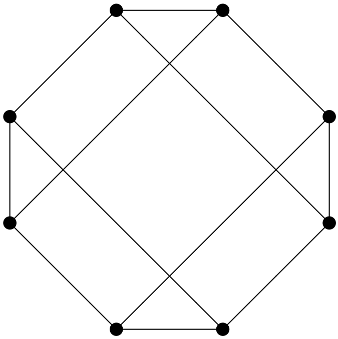

$\mathrm{HTG}(1, 8, 3)$

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We may assume that $\ell < n/2$ because $\ell = n/2$ corresponds to a circulant graph, and if $\ell > n/2$, we observe that $\mathrm{HTG}(1, n, \ell) \cong \mathrm{HTG}(1, n, n - \ell)$.

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Theorem. The honeycomb toroidal graph $\mathrm{HTG}(1, n, \ell)$, $\ell < n/2$, is $\mathcal{F}(1, 2)$-invariant if and only if $n \geq 18$ and none of the following congruences are satisfied:

- $(\ell + 1)^2/4 \equiv 1(mod\ n/2)$;
- $(\ell - 1)^2/4 \equiv 1(mod\ n/2)$;
- $(\ell^2 - 1)/4 \equiv -1(mod\ n/2)$;

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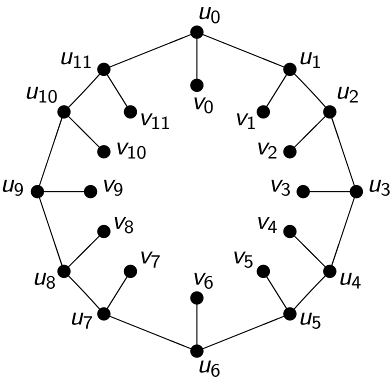
As an example illustrating the second theorem, consider the famous Heawood graph of order 14 which is $\mathrm{HTG}(1, 14, 5)$.

The automorphism group of the Heawood graph has order 336 so that it is far removed from being $\mathcal{F}(1, 2)$-invariant. Note that $(5^2 - 1)/4 = 6 \equiv -1 (\mathrm{mod}\ 7)$ which is the third condition in the theorem.

Two Cycles

We now consider $\mathcal{F}(1, 2)$-invariant graphs whose 2-factor is made up of two *n*-cycles. We shall find that an easy well-known family arises, but there is another family which surprised us in the sense we hadn't considered them before.

Two Cycles



A typical spanning subgraph of an $\mathcal{F}(1,2)$-invariant graph whose 2-factor consists of two n-cycles.

Generalized Petersen Graphs

We shall let C_1 denote the n-cycle $[u_0, u_1, \ldots, u_{n-1}, u_0]$ and the 1-factor consists of the edges $[u_i, v_i]$, $i = 0, 1, \ldots, n-1$.

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Because of the spanning subgraph pictured on the last slide, if f is an automorphism of an $\mathcal{F}(1, 2)$-invariant graph such that $f : C_1 \to C_1$, then the action of f on C_1 precisely determines the action of f on C_2. So it is enough to know the action of f on C_1.

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For the rest of the talk, X denotes a trivalent vertex-transitive graph of order $2n$ for which $\mathrm{Aut}(X)$ contains a subgroup acting $\mathcal{F}(1,2)$-invariantly on X such that the 2-factor consists of two n-cycles.

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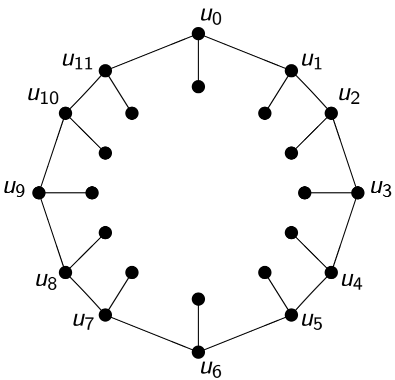

Let H be the subgroup of $\mathrm{Aut}(X)$ mapping C_1 to C_1. Then the restriction of H to C_1 must be a transitive subgroup of the dihedral group D_n. If H contains ρ, then the subgraph induced on $\{v_0, v_1, \ldots, v_{n-1}\}$ must be a circulant graph.

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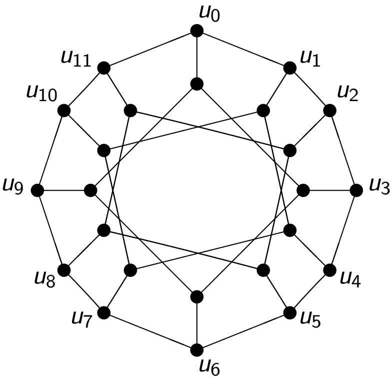

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Generalized Petersen Graphs

The preceding is an example of a generalized Petersen graph. The graph $\mathrm{GP}(n, k)$ is defined by starting with the initial spanning subgraph we had and then joining vertices v_i and v_{i+k} for all $i = 0, 1, \ldots, n-1$, where k is a fixed integer satisfying $0 < k < n$ and the arithmetic is done modulo n. The preceding picture is $\mathrm{GP}(12, 3)$.

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Generalized Petersen graphs have been studied since the 1960s. The following theorem was proved in 1971 by Frucht, Graver and Watkins.

Theorem. The generalized Petersen graph $\mathrm{GP}(n, k)$ is vertex-transitive if and only if $k^2 \equiv \pm 1 (\mathrm{mod}\ n)$.

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Theorem. The generalized Petersen graph $\mathrm{GP}(n, k)$ is $\mathcal{F}(1, 2)$-invariant if and only if $k^2 \equiv \pm 1 \pmod{n}$ with the exceptions of (n, k) in

$$\{(4, \pm 1), (5, \pm 2), (8, \pm 3), (10, \pm 3), (12, \pm 5) \text{ and } (24, \pm 5)\}.$$

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There are two chords incident with v_0. If you play with the possibilities, you will see that both chords must have their other ends of the form v_i, i odd, in order to have the subgraph induced on $\{v_0, v_1, \ldots, v_{n-1}\}$ connected. As we require this subgraph to be a spanning cycle, that condition is forced.

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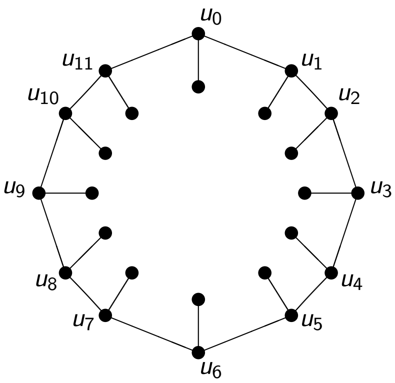

So we define $\mathcal{M}_O(n, a, b)$ to be the graph we obtain by starting with the graph shown and adding edges from v_i, i even, to v_{i+a} and v_{i+b}, where $a < b$ are both odd.

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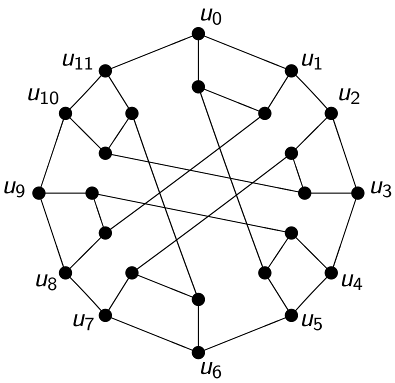

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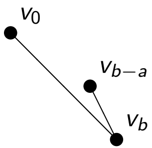

Starting the path at v_0, we see that two steps take us to v_{b-a} and $b-a$ is even. There are $n/2$ even subscripted vertices and we have travelled a jump of $(b-a)/2$ through them. Thus, we get a full cycle of of length n if $\gcd(n/2,(b-a)/2) = 1$ which is equivalent to the condition of the lemma.

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The preceding lemma is proved by observing that if the stabilizer of u_0 is not the identity, then the faithful extension of ρ is an automorphism of X. This implies that $b = n - a$ which contradicts an hypothesis.

Another Family

Theorem. If $X = \mathcal{M}_O(n, a, b)$, where $\gcd(b-a, n/2) = 1$, then $\mathrm{Aut}(X)$ contains a subgroup G acting $\mathcal{F}(1,2)$-invariantly on X if and only if $(b-a)^2/2 \equiv 2 (\mathrm{mod}\ n)$ and at least one of the following congruences holds:
(1) $a + (a-1)(a-b)/2 \equiv 1 (\mathrm{mod}\ n)$ and
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This theorem is proved by working with the automorphism σ in G that maps u_0 to v_0. It must be an involution and maps the edge $[u_0, u_1]$ to either $[v_0, v_a]$ or $[v_0, v_b]$. Once the image of $[u_0, u_1]$ is known, σ is completely determined and the congruences are forced.

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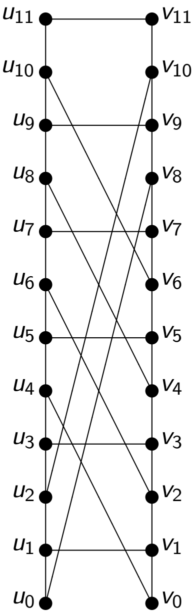
We find it more convenient to work with honeycomb toroidal graphs and choose to do so for the subfamily for which $b - a = 2$.

A Dense Subfamily



The honeycomb toroidal graph $\mathrm{HTG}(2, 12, 4)$.

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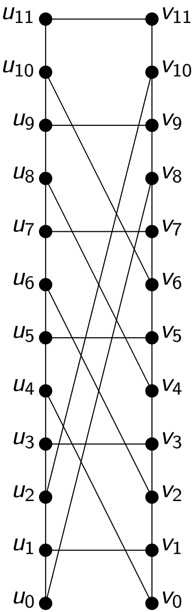

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From this example we can see that the following permutations are automorphisms of $\mathrm{HTG}(2, n, \ell)$.

$\varphi : u_i \to u_{i+2}$ and $\varphi : v_j \to v_{j+2}$

$\pi : u_i \to v_{i+1}$ and $\pi : v_j \to u_{j+\ell+1}$

ω interchanges u_{1+i} and v_{1-i}.

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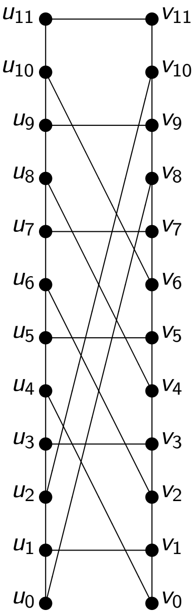

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ω interchanges u_{1+i} and v_{1-i}.

It is easy to see that φ and π commute, and that ω conjugates both to their inverses. Hence, $\langle \varphi, \pi, \omega \rangle$ is a generalized dihedral group and acts regularly on $\mathrm{HTG}(2, n, \ell)$.

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Applying his result to the special case of honeycomb toroidal graphs of the form $\mathrm{HTG}(2, n, \ell)$, we have the following theorem.

Theorem. The honeycomb toroidal graph $\mathrm{HTG}(2, n, \ell)$ is $\mathcal{F}(1, 2)$-invariant if and only if none of the following hold:
(i) $\gcd(n, \ell + 2) = 4$ and $4n | (\ell^2 + 4\ell - 12)$;
(ii) $\gcd(n, \ell - 2) = 4$ and $4n | (\ell^2 - 4\ell - 12)$; and
(iii) $\gcd(n, \ell + 2) = 4 = \gcd(n, \ell - 2)$ and $4n | (\ell^2 + 12)$.

Sparse Families

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Suppose that $(b-1)^2/2 \equiv 2 (\text{mod } n)$. The congruence $a + (a-1)(a-b)/2 \equiv 1 (\text{mod } n)$ is always satisfied for $a = 1$. Thus, $\mathcal{M}_O(n, 1, b)$ is a candidate to be $\mathcal{F}(1,2)$-invariant.

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Let's consider some local structure when $a = 1$.

Sparse Families

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Moreover, when $a = 1$, it is easy to see that the only 4-cycles are $[u_i, u_{i+1}, v_{i+1}, v_i, u_i]$, i even, so that they form a 2-factor. Thus, the vertex sets of these 4-cycles form blocks of imprimitivity for $\mathrm{Aut}(X)$.

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Removing the 4-cycles leaves a perfect matching M and it's clear that the vertex sets of the edges of M also form a block system.

Sparse Families

However, there is an even more interesting block system which we now describe. Form an auxiliary graph Y by taking the edges of M and adding the diameter edges of the 4-cycles. We see that Y is regular of valency 2.

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As each vertex of an anchor chain is incident with an edge of M and a diameter edge of a 4-cycle, if a vertex of an anchor chain is fixed by an automorphism of X, then every vertex of the anchor chain is fixed.

Sparse Families

Working with anchor chains and the stabilizer of u_0, it is straightforward to prove the following theorem.

Theorem. If $X = \mathcal{M}_O(n, 1, b)$ satisfies the congruence conditions for vertex transitivity given earlier and $3 < b < n - a$, then X is $\mathcal{F}(1, 2)$-invariant if and only if the parameters do not satisfy $n \equiv 0 \pmod 8$ and $b = (n + 6)/2$.

Sparse Families

To eliminate cases which already are settled, we now assume that the congruence conditions for vertex transitivity are satisfied and $1 < a < b - 2 < n - a - 2$. We say that $X = \mathcal{M}_O(n, a, b)$ is feasible when this is the situation.

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Given a feasible $\mathcal{M}_O(n, a, b)$, we saw earlier that its automorphism group contains a regular subgroup G. In fact, the graph is $\mathcal{F}(1, 2)$-invariant when there are no additional automorphisms.

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Let $X = \mathcal{M}_O(n, a, b)$ be feasible and G the aforementioned regular group of automorphisms.

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Now G has three orbits acting on the edges of X: The orbit $\mathcal{B}$ containing the edge $[u_0, u_1]$, the orbit $\mathcal{G}$ containing the edge $[u_1, u_2]$, and the orbit $\mathcal{R}$ containing the edge $[u_0, v_0]$ (the spokes).

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If $X = \mathcal{M}_O(n, a, b)$ is feasible, then it's clear that its orbits are one of

1. $\mathcal{B}, \mathcal{G}$ and $\mathcal{R}$;
2. $\mathcal{B} \cup \mathcal{R}$ and $\mathcal{G}$;
3. $\mathcal{G} \cup \mathcal{R}$ and $\mathcal{B}$; or
4. $\mathcal{B} \cup \mathcal{G} \cup \mathcal{R}$.

Sparse Families

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The fourth option corresponds to X being arc-transitive. Because ρ^2 is an automorphism composed of a product of four disjoint cycle of length $n/2$, X is a tetracirculant graph. Arc-transitive tetracirculant graphs were characterized by Frelih and Kutnar in 2013.

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Using their result, the following holds.

Proposition. If $X = \mathcal{M}_O(n, a, b)$ is feasible, then $\mathrm{Aut}(X)$ contains an automorphism fixing u_0 and cyclically permuting $\mathcal{B}, \mathcal{G}$ and $\mathcal{R}$ if and only if $n \equiv 8 (\mathrm{mod}\ 16)$ and either:
(1) $a = 4a_0 + 1 < n/4 - 1$, where a_0 is odd and satisfies $8(a_0^2 + a_0 + 1) = 0$ in $\mathbb{Z}_n$, and $b = n/2 + a + 2$; or
(2) $b = 4b_0 + 1 < n/4 - 1$, where $n/2 < b < (3n-4)/4, b_0$ is odd and satisfies $8(b_0^2 + b_0 + 1) = 0$ in $\mathbb{Z}_n$, and $a = b - n/2 + a + 2$.

Sparse Families

We are left with the second and third options for the orbits of G acting on the edges. They require essentially the same arguments to determine the graphs which allow non-identity automorphisms in the stabilizer which produce the given edge orbits.

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The subgraph composed of the edges of $\mathcal{B} \cup \mathcal{R}$ is a 2-factor. It can be shown that if the components of this 2-factor are not 8-cycles, then the stabilizer of u_0 is either the identity or has order 2.

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So the 8-cycle $[u_0, v_0, v_b, u_b, u_{b-1}, v_{b-1}, v_1, u_1, u_0]$ and its images play a major role.

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The next slides summarize the main result.

Summary

We now list the $\mathcal{F}(1,2)$-invariant graphs where the 2-factor has two cycles.

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(1) All generalized Petersen graphs $\mathrm{GP}(n,k)$ for which $k^2 \equiv \pm 1 (\mathrm{mod}\ n)$ with the exception of those for which (n,k) is in the following list:

$$(4,\pm 1),(5,\pm 2),(8,\pm 3),(10,\pm 3),(12,\pm 5) \text{ and } (24,\pm 5).$$

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(2) The honeycomb toroidal graphs $\mathrm{HTG}(2,n,\pm\ell), 0<\ell<n/2$, with the exception of those satisfying any one of the following conditions:

1. $\gcd(n,\ell+2)=4$ and $4n|(\ell^2+4\ell-12)$;
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(3). The graphs $\mathcal{M}_O(n,1,b)$, $3<b<n-2$ if and only if $\gcd(n,b-1)=2,(b-1)^2/2\equiv 2(\mathrm{mod}\ n)$ and the parameters do not satisfy $n\equiv 0(\mathrm{mod}\ 8)$, $n>8$ and $b=(n+6)/2$.

Summary

The feasible graphs $\mathcal{M}_O(n, a, b)$ if and only if the parameter set satisfies none of the following:

1. $n \equiv 8 \pmod{16}$, $a = 4a_0 + 1 < n/4 - 1$, a_0 odd, such that $8(a_0^2 + a_0 + 1) = 0$ in $\mathbb{Z}_n$, and $b = n/2 + a + 2$;
2. $n \equiv 8 \pmod{16}$, $b = 4b_0 + 1$, b_0 odd, such that $8(b_0^2 + b_0 + 1) = 0$ in $\mathbb{Z}_n$, $n/2 < b < (3n - 4)/4$ and $a = b - n/2 + 2$;
3. $n \equiv 48 \pmod{96}$ and either $a = n/4 + 3$ and $b = n/2 + 1$, or $a = n/4 - 3$ and $b = n/2 - 1$;
4. $8|n$, $b = 4b_0 + 3 < (3n + 4)/4$, with either $b_0 > 0$ and even, $4(b_0 + 1)^2 \equiv 4 \pmod{n}$ and $a = b - n/2 - 2$; or b_0 odd, $4b_0^2 \equiv 4 \pmod{n}$ and $a = b - n/2 + 2$; and
5. $8|n$ and either $a = 4a_0 + 3 < (n + 4)/4$, a_0 even, $4(a_0 + 1)^2 \equiv 4 (\mathrm{mod}\ )$ and $b = n/2 + a - 2$; or $b = 4b_0 + 1 < (3n - 4)/4$, b_0 odd, $4b_0^2 \equiv 4 \pmod{n}$ and $a = b - n/2 + 2$.

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The first question then is "are there examples other than honeycomb toroidal graphs?" We have barely considered this question, but we do know the answer is yes, that is, there are other families of graphs that are not honeycomb toroidal graphs and they look interesting.

Thank You