Overview and Recent Contributions to the Evolution of Regular Polyhedra and Polytopes

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Some history

Theaetetus (416 - 368 BC), a contemporary of Plato, gave a mathematical description of all five Platonic solids and may have been responsible for the first known proof that no other convex regular polyhedra exist. Book XIII, the climax of the Euclid's *Elements*, dealing with Platonic solids is derived from the work of Theaetetus.

Archimedes (287 - 212 BC) enumerated semi-regular convex polyhedra and although no work of his on polyhedra survived it is mentioned in the work of Pappus of Alexandria.

...

...

Kepler, Poinsot, Cauchy, ...

Schläfli (1814 - 1895) extends the concept of a poyhedron to higher dimensions. Although his work was completed between 1850 and 1852, it was was published posthumously in 1901. Schläfli introduced the concept of higher-dimensional polytopes, which he called polyschemes. He proved that there are exactly six regular polytopes in four dimensions and only three in dimensions higher than four. He introduced the notion of what we now call Schläfli symbol.

Between 1890 and 1895 Stott working alone without any scientific contact rediscovered the six regular polytopes in dimension 4. She continues her work in 4 dimensions in collaboration with Schoute between 1985 and 2013.

Coxeter (1907 - 2003) is responsible for revival of interest in polyhedra and polytopes and the most comprehensive treatment of the subject. He collaborated with Stott between 1930 and 1940. His investigations of symmetries of these structures led him to the enumeration of reflection groups.

Grünbaum (1929 - 2018) in 1977 makes a contribution of great significance with the publication "Regular polyhedra - Old and new".

Enumeration of regular polyhedra

Regular polyhedra with convex faces - FINITE

Platonic Solids

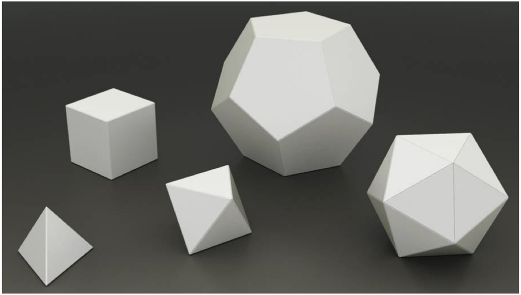


Photo credit: Rinus Roelofs

Schläfli type: $\{3,3\}$ $\{4,3\}$ $\{3,4\}$ $\{5,3\}$ $\{3,5\}$

Note: The vertex-figures are also convex polygons.

Regular polyhedra with convex faces - INFINITE

Regular Sponges - Petrie-Coxeter polyhedra

(Discovered in 1926 and related to $\{4, 3, 4\}$, a cubical tessellation of E^3.)

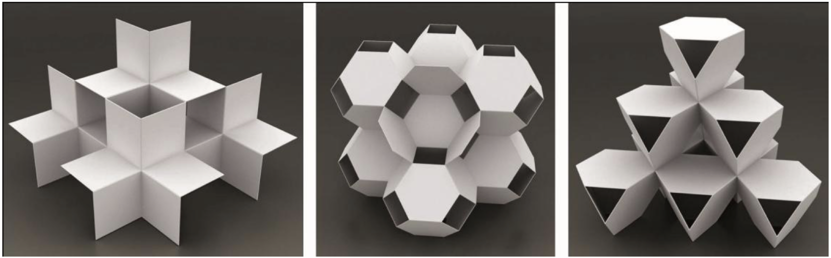
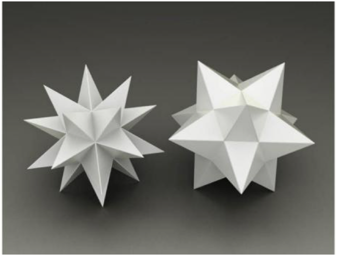


Photo credit: Rinus Roelofs

Schläfli type:	$\{4, 6_s\}$	$\{6, 4_s\}$	$\{6, 6_s\}$

Note: The faces are convex polygons, but the vertex-figures are skew polygons!

Regular polyhedra with non-convex (finite, planar) faces or vertex-figures



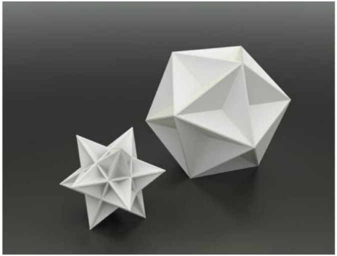


Photo credit: Rinus Roelofs

Schläfli type: $\{5/2, 3\}$ $\{5/2, 5\}$ $\{3, 5/2\}$ $\{5, 5/2\}$

The two stellated dodecahedra (on the left) were first recognized as regular by Kepler in 1619.

In 1809, Poinsot rediscovered Kepler's figures and discovered the great icosahedron and great dodecahedron.

In 1812 Cauchy proved that the list is complete.

Regular polyhedra in modern theory

Grünbaum (1929 - 2018) extended the definition of regular polyhedra to include non-planar faces.

"The Original Sin in the theory of polyhedra goes back to Euclid, and through Kepler, Poinsot, Cauchy, and many others continues to afflict all work on this topic.

...

*The writers failed to define what are the 'polyhedra' among which they are finding the regular ones."**

* B. Grünbaum "Polyhedra with hollow faces" in *POLYTOPES: Abstract, Convex and Computational*, NATO ASI series, 1994.

Regular polyhedra with non-planar (finite) faces

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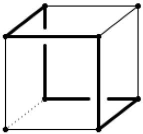

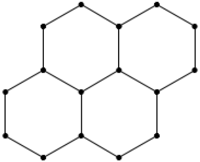

$\{6_s, 3\} \Longleftrightarrow$ Petrial of a cube $\{4, 3\}$

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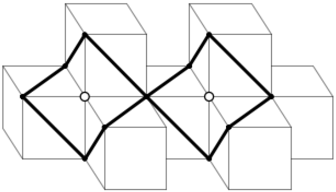
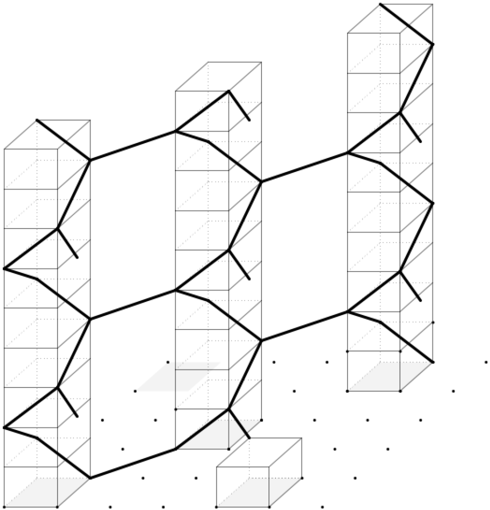


$\{6_s, 6\} \Longleftrightarrow$ one half of the vertex-figures of a Petrie-Coxeter poyhedron $\{4, 6_s\}$

Regular polyhedra with infinite faces

Grünbaum-Dress polyhedron

$\{\infty, 3\}_{[4]}$



Geometric Polyhedra

A geometric polyhedron is a discrete faithful realization of a (non-degenerate) map in E^3

$$\begin{array}{lcl} \text{vertex} & \mapsto & \text{point} \\ \text{edge} & \mapsto & \text{line segment} \\ \text{face} & \mapsto & \text{finite polygon or apeirogon} \end{array}$$

Note: Each compact set meets only finitely many faces and the degree of each vertex is finite.

A polyhedron in E^3 is said to be geometrically regular if its symmetry group (the group of isometries keeping the polyhedron invariant) is transitive on the set of its flags.

Note: Faces of geometrically regular polyhedra must be regular polygons.

One approach to Grünbaum's classification is through such realizations.

Classification of Geometrically Regular Polyhedra

(Grünbaum-Dress 1985):

Platonic solids	$\{3,3\}$ $\{3,4\}$ $\{4,3\}$ $\{3,5\}$ $\{5,3\}$	5
Kepler-Poinsot polyhedra	$\{3,5/2\}$ $\{5/2,3\}$ $\{5,5/2\}$ $\{5/2,5\}$	4
Petrials of these	$\cdots$	9
Regular tessellations of E^2	$\{4,4\}$ $\{3,6\}$ $\{6,3\}$	3
Blends of these with segments	$\cdots$	3
Blends of these with $\{\infty\}$	$\cdots$	3
Petrials of these	$\cdots$	9
Petrie-Coxeter polyhedra	$\{4,6\|4\}$ $\{6,4\|4\}$ $\{6,6\|3\}$	3
Grünbaum-Dress polyhedra		9
		—
		48

18 finite polyhedra
6 planar polyhedra
24 infinite 3-dimensional polyhedra

Geometrically chiral polyhedra

A polyhedron in E^3 is said to be geometrically chiral if its symmetry group has two orbits on the set of flags with adjecent fags in distinct orbits.

Note: A geometrically chiral polyhedron is either a chiral or a regular abstract polyhedron.

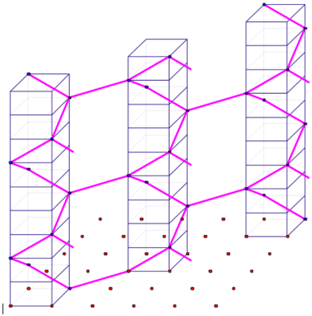
Theorem: Finite chiral (geometric) polyhedra do not exist.

Theorem (Schulte 2005): Discrete chiral polyhedra can be classified in the following six families.

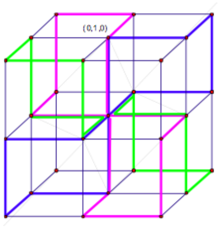
Finite faced polyhedra:	$\{6,6\}_{[a,b]}$	$\{4,6\}_{[a,b]}$	$\{6,4\}_{[a,b]}$
Infinite faced polyhedra:	$\{\infty,3\}_{[3]}$	$\{\infty,3\}_{[4]}$	$\{\infty,4\}_{[3]}$

Theorem (Pellicer, AIW 2010): Chiral polyhedra with finite faces are abstract chiral polyhedra. The chiral polyhedra with infinite faces are regular abstract polyhedra.

Examples of geometrically chiral polyhedra



$\{\infty, 3\}_{[4]}$



$\{6, 6\}_{[1,0]}$

Geometrically uniform polyhedra

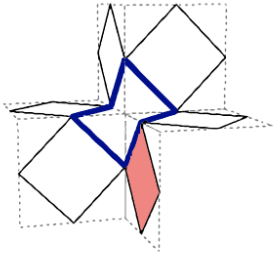
A geometric polyhedron P is defined to be *uniform* if P has regular faces and its symmetry group acts transitively on the vertices of P.

Finite uniform polyhedra with planar faces had been enumerated by Coxeter, Longuet-Higgins and Miller (1954). No classification to date is known for non-planar faces, however a number of examples have been constructed by Abigale Williams.

Example

with vertex type $(4_c.6_s.4_c.6_s)$

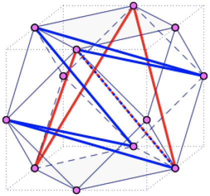
An infinite uniform polyhedron with hexagonal skew faces derived from a Petrie-Coxeter polyhedron using Wythoff construction.



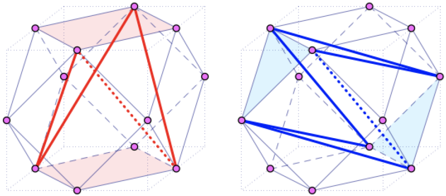
Example

Non-Wythoffian, uniform polyhedron

with vertex type $(4_s.6_s.4_s.6_s)$



Construction (due to A. Williams) of a finite uniform polyhedron with 10 vertices, 24 edges and 10 faces (six 4_s and four 6_s).



Realizations of finite regular polytopes

Grünbaum classified finite regular geometric polyhedra in E^3 into 3 classes: 5 Platonic solids, 4 star-polyhedra, and 9 of their Petrials.

A different approach: Given a finite regular abstract polytope realize it as a symmetric object in a euclidean n-space.

Example: The abstract polygon $\{6\}$ can be realized as

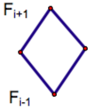
- convex hexagon $\{6_c\}$ in E^2,
- regular simple skew hexagon $\{6_s\}$ (Petrie polygon of a cube) in E^3,
- self-intersecting skew hexagon (inscribed in a triangular prism) in E^3,
- hexagon containing all vertices of regular simplex in E^5.

Goal: Describe all such realizations for a specific polytope.

Abstract Polytopes

An *abstract polytope P of rank n, or an n-polytope* is a poset, whose elements are called *faces*, with strictly monotone rank function with range $\{-1, 0, 1, \ldots, n\}$ satisfying the following properties.

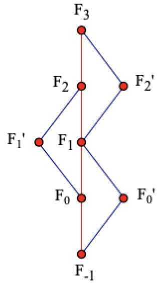
- P has a unique minimal face F_{-1} and a unique maximal face F_n.
- The maximal chains, called *flags*, of P contain exactly $n+2$ faces.
- P is strongly flag-connected.
- P satisfies a homogeneity property.



Regular abstract polytopes

Abstract polytope P is said to be *regular* if its group of automorphisms $Aut(P)$ is transitive on the flags of P.

$\Longrightarrow$ $Aut(P)$ is generated by involutions (determined by the "base" flag).

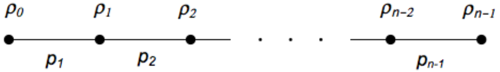


ρ_{n-1}

ρ_1

ρ_0

$C-$diagram:



Given that P is a regular $n-$polytope and Φ one of its flags, $Aut(P)$ is generated by the distinguished generators ρ_i, $i = 0, \ldots, n-1$, that interchange Φ with its $i-$adjecent flag Φ^i and satisfy the relations implicit in the string Coxeter graph associated with the string Coxeter group $[p_1, \ldots, p_{n-1}]$.

$\Rightarrow$ Regular polytopes can be assigned a Schläfli type $\{p_1, \ldots, p_{n-1}\}$.

The generators of the automorphism group of an abstract polytope satisfy an intersection property *IP*:

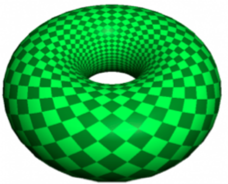
$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle, \quad \forall I, J \subseteq \{0, \ldots, n-1\}.$$

Characterization of groups of regular abstract polytopes

A quotient of a string Coxeter group $[p_1, \ldots, p_{n-1}]$ with generators that satisfy the intersection property IP is called a $C-$group.

Theorem (Schulte, 1982): Given a $C-$group one can construct a regular polytope having this group as its automorphism group.

Example: From a quotient of the Coxeter group $[4, 4]$ by a translation subgroup one can construct regular polytope of rank 3 (a regular map on torus of Schläfli type $\{4, 4\}$).



Realizations of finite abstract regular polytopes

Let $\mathcal{P}$ be a finite regular abstract n-polytope, $\mathcal{P}_j$ be the set of its j-faces. Following McMullen, we define a realization of $\mathcal{P}$ consisting of

- a set V_0 of points in a euclidean space E, together with a surjection $\beta = \beta_0 : \mathcal{P}_0 \to V_0$, such that
- each automorphism of $\mathcal{P}$ induces an isometric permutation of V_0.

For $j = 1, \ldots, n$, β_0 recursively induces a surjection $\beta_j : \mathcal{P}_j \to V_j$, with $V_j \subseteq 2^{V_{j-1}}$ given by $F\beta_j = \{G\beta_{j-1} | G \in \mathcal{P}_{j-1} \text{ and } G \leq F\}$ for $F \in \mathcal{P}_j$.

Then $P := V_n$ encapsulates all the structure of $\mathcal{P}$. We refer to it as realization of $\mathcal{P}$ and call it a (geometric) polytope.

We say that the realization P (or β) is faithful if each β_j is a bijection.

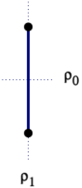
We say that the realization is pure if its symmetry group is irreducible.

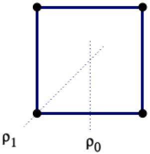
A (faithful) realization of a regular polytope $\mathcal{P}$ induces a (faithful) realization of each section of $\mathcal{P}$. For finite polytopes faithful realizations always exist.

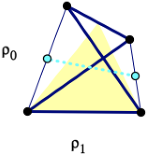
When $G(P)$ act reducibly on E then in a natural way P is congruent to a blend of lower dimensional realizations. Each realization is a blend of pure realizations.

Example: $\{4\}$ can be realized in

E^1 as a segment $\{\}$ E^2 as a square $\{4_c\}$ E^3 as $\{4_s\} = \{\}\#\{4_c\}$



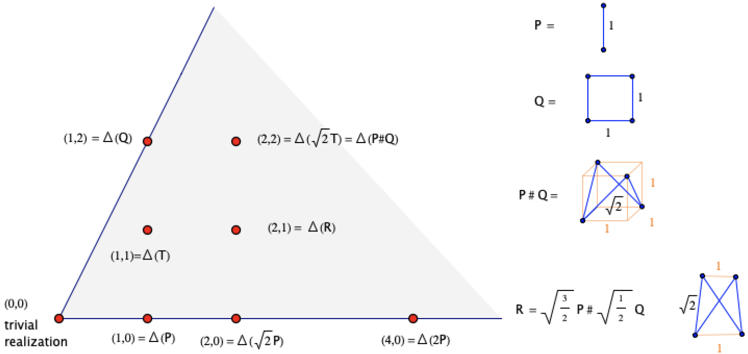




Realizations cone

The congruence classes of realizations have the structure of a convex r-dimensional cone, where r is the number of diagonal classes in $\mathcal{P}$ (a diagonal is an unordered pair of distinct vertices of $\mathcal{P}_0$).

Example:



Examples

$\{5\}$ has two pure, faithful realizations: $\{5\}$ (2-dim) and $\{\frac{5}{2}\}$ (2-dim) (note that $\{5\}\#\{\frac{5}{2}\}$ is a 4-dimensional simplex realization).

$\{6\}$ has three pure realizations: $\{\}$ (1-dim, deg), $\{3\}$ (2-dim, deg), and $\{6_c\}$ (2-dim, faithful) (note that $\{\}\#\{6_c\} = \{6_s\}$, $\{\}\#\{3\}$ is a self-intersecting skew hexagon, and $\{\}\#\{3\}\#\{6_s\}$ is a 5-dimensional simplex realization).

$\{3,3\}$ has a unique (simplex) realization.

$\{3,4\}$ has two pure realizations: octahedron $\{3,4\}$ (3-dim, faithful) and $\{3\}$ (2-dim, deg) (their blend is 5-dimensional simplex realization).

$\{4,3\}$ has three pure realizations: $\{\}$ (1-dim, deg), cube $\{4,3\}$ (3-dim, faithful), and $\{3,3\}^\pi$ (3-dim, deg realization that is a faithful realization of the hemi-cube $\{4,3\}_3$).

$\{3,5\}$ has three pure realizations: icosahedron $\{3,5\}$ (3-dim, faithful), great icosahedron $\{3,\frac{5}{2}\}$ (3-dim, faithful), and a 5-dimensional degenerate realization having vertices of a 5-dimensional simplex (that is a faithful realization of the hemi-icosahedron $\{3,5\}_5$).

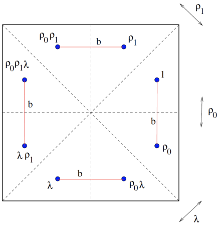
$\{5,3\}$ has five pure realizations: two faithful in dimension 3, one faithful in dimension 4, and two degenerate in dimensions 4 and 5.

Note: Only 3-dim faithful realizations above are regular polyhedra as defined by Grünbaum.

Realizations of regular toroids

In 1999 - 2000, with Barry Monson, we explicitly determine and describe the pure realizations of finite regular toroidal polyhedra (toroidal maps).

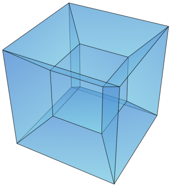
Realizations of $\{4,4\}$ For a fixed b, we start with group K which is the direct product of 4 dihedral groups each of order $2b$ (so K has 8 involutory generators indexed by the elements of $B_2 = \langle \rho_0, \rho_1 \rangle$ and order $(2b)^4$).



We extend K by twisting operations (on the group K) to get $K^{'} = K \rtimes B_2$ and faithfully represent $K^{'}$ as a group of orthogonal transformations in the 8-dimensional euclidean space.

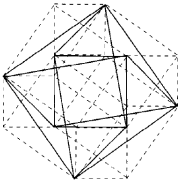
Example $\{4,4\}_{(4,0)}$ has six inequivalent pure realizations. One each of 0, 1, and 2-dimensions, and three 4-dimensional realizations:

- a faithful realization consisting of the 16 vertices, the 32 edges of the hypercube $\{4,3,3\}$, and sixteen squares belonging to a "horizontal" belt of four cubes;



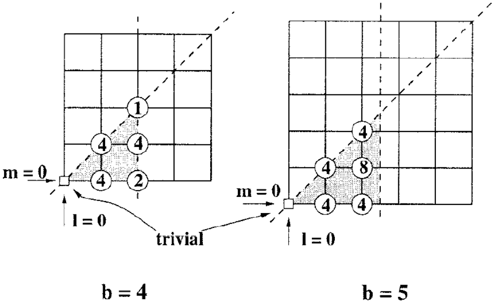
- a faithful realization consisting of 16 vertices of the hypercube, the 32 edges that are the body diagonals of the eight cubical facets of the hypercube, and the 16 skew faces (each inscribed in a belt of four cubical facets);

- a non-faithful (2:1 collapse) realization consisting of 8 alternate vertices of the hypercube, 16 edges that are the diagonals of the square faces of the hypercube connecting selected vertices, and the 8 skew faces.



In general, let $\mathcal{P} = \{4,4\}_{(b,0)}$ with $b \geq 2$. For each $0 \leq m \leq l \leq \frac{b}{2}$ there are distinct pure realizations $P_{m,l}$ of $\mathcal{P}$.

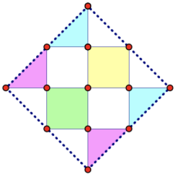
The dimensions of the realizations can be neatly be encoded in a picture of $\mathcal{P}$ as toroidal map:

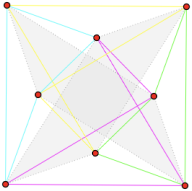


Inequivalent pure realizations are indexed by the vertices in the fundamental region for symmetry group of the grid. The dimension of each non-trivial pure realization $P_{m,l}$ equals the size of the orbit of the corresponding grid vertex, under the dihedral group generated by the two grid symmetries and toroidal identifications.

Since $\{4,4\}_{(2b,0)}$ doubly covers $\{4,4\}_{(b,b)}$ (which belongs to the other family of regular toroidal maps), every realization of the toroid $\{4,4\}_{(b,b)}$ is some pure realization of $\{4,4\}_{(2b,0)}$.

Example: Recall, the non-faithful realization of $\{4,4\}_{(4,0)}$ onto the alternate vertices of the hypercube. These 8 vertices are vertices of an inscribed (regular) cross-polytope providing a faithful realization of $\{4,4\}_{(2,2)}$.





The 16 edges of $\{4,4\}_{(2,2)}$ are realized as just those edges of the cross-polytope which remain after removing those in two orthogonal equatorial squares (indicated in grey).

Incidence Systems

We next extend the concept of a polytope to a more general structure of a hypertope.

An incidence system $\Gamma := (X, *, t, I)$ is a 4-tuple such that

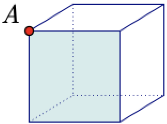
- X is a set whose elements are called the elements of Γ;
- I is a finite set whose elements are called the types of Γ;
- $t : X \to I$ is a type function, associating to each element $x \in X$ of Γ a type $t(x) \in I$;
- $*$ is a binary relation on X called incidence, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

The rank of Γ is the cardinality of I.

Examples: cube, cube with an edge appended at a vertex, skeletal cube...

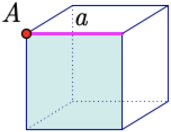
Incidence geometries

A flag is a set of pairwise incident elements of Γ.



The type of a flag F is $\{t(x) : x \in F\}$.

A chamber is a flag of type I.

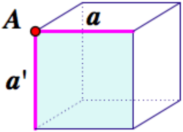


An incidence system Γ is a geometry (or incidence geometry) if every flag of Γ is contained in a chamber.

Examples: cube, skeletal cube.

Thin geometries

A geometry Γ is called thin if for each $i \in I$ any flag of type $I \setminus \{i\}$ is contained in exactly two chambers.



The homogeneity condition in the definition of abstract polytopes guaranties that abstract polytopes are thin geometries.

Skeletal cube is not a thin geometry.

Non-degenerate maps and hypermaps are examples of thin geometries.

Residues and automorphisms

Let $\Gamma := (X, *, t, I)$ be an incidence geometry and F a flag of Γ. The residue of F in Γ is the incidence geometry $\Gamma_F := (X_F, *_F, t_F, I_F)$ where

- $X_F := \{x \in X : x * F, x \notin F\}$;
- $I_F := I \setminus t(F)$;
- t_F and $*_F$ are the restrictions of t and $*$ to X_F and I_F.

An automorphism of $\Gamma := (X, *, t, I)$ is a mapping $\alpha : X \mapsto X$ such that for all $x, y \in X$

- α is a bijection on X (inducing a bijection on I);
- $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

An automorphism is type preserving when for each $x \in X$, $t(\alpha(x)) = t(x)$. The set of all type preserving automorphism of Γ is denoted by $Aut_I(\Gamma)$.

Γ is chamber transitive if $Aut_I(\Gamma)$ is transitive on the set of chambers of Γ.

Hypertopes

A hypertope is a thin incidence geometry that is strongly chamber connected (SCC). (Or, residually connected as commonly used in the terminology of incidence geometries).

Examples: abstract polytopes, non-degenerate hypermaps ...

A hypertope Γ is said to be regular if $Aut_I(\Gamma)$ has one orbit on the chambers of Γ.

Groups of regular hypertopes

Let Γ be a regular hypertope and Φ one of its chambers. Then for each $i \in I$ there exists and involutory type-preserving automorphism ρ_i that interchanges Φ with its i-adjacent chamber Φ^i.

$Aut_I(\Gamma)$ is generated by the distinguished generators $\{\rho_0, \rho_1, \ldots, \rho_{n-1}\}$, where $n = |I|$, which satisfy

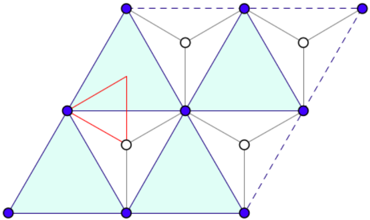
- the relations implicit in the $C-$diagram, the complete graph on n vertices whose vertices are labeled by the generators and the edges between vertices labelled with ρ_i and ρ_j labeled by $o(\rho_i \rho_j)$ (with the usual convention of omitting the edges labeled by 2);
- and the intersection property IP

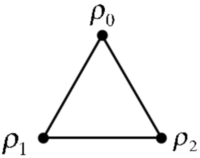
$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle, \quad \forall I, J \subseteq \{0, \ldots, n-1\}.$$

$C-$Groups

A pair (G, R), where G is a group and $R = \{\rho_0, \ldots, \rho_{n-1}\}$ its generating set of involutions that satisfy the IP, is called a $C-$group.

The group $\langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1\rho_2\rho_1)^2 = 1 \rangle$ with the triangular $C-$diagram is the group of automorphisms of the hypermap $(3,3,3)_{(2,0)}$.





A hypertope is said to be proper when the Coxeter diagram of its type preserving automorphism group is not linear.

Coset geometries

Construction of an incidence geometry from a group (Tits, 1956):

Let G be a group and $(G_i)_{i \in I}$ a finite family of subgroups of G. With X, $*$ and t defined as

- X is the set of all cosets $G_i g$, $g \in G$, $i \in I$;
- $t : X \to I$ defined by $t(G_i g) = i$;
- $G_i g_1 * G_j g_2$ if a and only if $G_i g_1 \cap G_j g_2 \neq \emptyset$;

$\Gamma := (X, *, t, I)$ is an incidence system.

Question: When is such an incidence geometry a hypertope?

Constraction of a regular hypertopes from a group

Theorem (Fernandes, Leemans and AIW, 2014) Given that $(G, \{\rho_0, \rho_1, \rho_2\})$ is a $C-$group of rank 3, the coset geometry $\Gamma(G, (\langle \rho_1, \rho_2 \rangle, \langle \rho_0, \rho_2 \rangle, \langle \rho_0, \rho_1 \rangle))$ is thin if and only if G acts faithfully on Γ and is transitive on chambers. Moreover, if it is thin it is strongly chamber-connected.

Unfortunately in higher ranks thinness need not suffice!

Theorem (Fernandes, Leemans and AIW, 2014) Given that $(G, S = \{\rho_0, \rho_1, \ldots, \rho_{n-1}\})$ is a $C-$group of rank n, the coset geometry $\Gamma := \Gamma(G, (G_i)_{i \in I})$ with $G_i := \langle \rho_j \mid \rho_j \in S, j \in I \setminus \{i\} \rangle$ for all $i \in I := \{0, 1, \ldots, n-1\}$, if Γ is flag transitive, then Γ is regular incidence geometry (it is thin, SCC and regular giving a regular hypertope).

Universal hypertopes

Every Coxeter group is a type-preserving automorphism group of a regular hypertope called the universal hypertope (associated with the Coxeter group).

The type-preserving automorphism group of every regular hypertope H, is a quotient of a Coxeter group $\mathcal{C}$.

The universal hypertope associated with the Coxeter group $\mathcal{C}$ is then called the universal cover of the hypertope H and the Coxeter diagram of H is the diagram of its universal cover.

A regular hypertope H with the universal cover whose Coxeter group is $\mathcal{C}$ is said to be of type $\mathcal{C}$ (where, for convenience, $\mathcal{C}$ names both Coxeter group and its diagram).

Hypertopes of locally spherical type

A regular hypertope is said to be of spherical type if its Coxeter diagram is a diagram of a finite irreducible Coxeter group. It is said to be spherical if its Coxeter diagram is a union of diagrams of finite irreducible Coxeter groups.

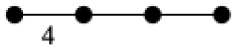
A projective regular hypertope is a hypertope obtained by factoring a spherical regular hypertope by the central symmetry (provided it exists).

A locally spherical regular hypertope is a hypertope whose (all) proper residues are spherical.

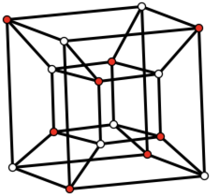
Theorem Let H be a regular hypertope of spherical type. Then
(1) H is either a spherical or a projective hypertope;
(2) H is locally spherical.

Examples

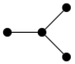
Hypertopes of type B_n:



hypercube (spherical)
hemi-hypercube (projective).

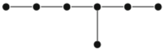


Hypertope of type D_4:



Type 0: red vertices of the hypercube
Type 1: white vertices of the hypercube
Type 2: faces of the hypercube with bi-coloured vertices
Type 3: cubes of the hypercube with bi-coloured vertices

The following is the complete list of locally spherical regular hypertopes of spherical type.

Diagram	Group	Order	Universal hypertope	Projective hypertope
A_n $(n \geq 1)$ 	$[3^{n-1}]$	$(n+1)!$	$\{3^{n-1}\}$	–
D_n $(n \geq 4)$ 	$[3^{n-3,1,1}]$	$2^{n-1}n!$	$\left\{3^{n-3}, \begin{smallmatrix}3\\3\end{smallmatrix}\right\}$	–
B_n $(n \geq 3)$ 	$[3^{n-2}, 4]$	$2^n n!$	$\{3^{n-2}, 4\}$	$\{3^{n-2}, 4\}_n$
I_2^p $(p \geq 3)$ 	$[p]$	$2p$	$\{p\}$	–
E_6 	$[3^{2,2,1}]$	$12 \cdot 6!$	$\{2_{2,1}\}$	–
E_7 	$[3^{3,2,1}]$	$8 \cdot 9!$	$\{3_{2,1}\}$	$\{3_{2,1}\}_9$
E_8 	$[3^{4,2,1}]$	$192 \cdot 10!$	$\{4_{2,1}\}$	$\{4_{2,1}\}_{15}$
F_4 	$[3, 4, 3]$	1152	$\{3, 4, 3\}$	$\{3, 4, 3\}_6$
H_3 	$[3, 5]$	120	$\{3, 5\}$	$\{3, 5\}_5$
H_4 	$[3, 3, 5]$	14400	$\{3, 3, 5\}$	$\{3, 3, 5\}_{15}$

Locally spherical hypertopes of euclidean type

A regular hypertope of euclidean type is a hypertope whose Coxeter diagram is a diagram of an irreducible Coxeter group of euclidean type.

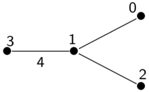
$\Rightarrow$ Proper residues of regular hypertopes of euclidean type are either spherical or projective.

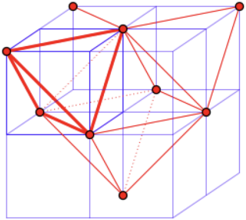
A regular toroidal hypertope of rank $n+1$ is a quotient of a regular universal hypertope of rank $n+1$ of euclidean type by a (normal) subgroup generated by n independent translations (in E^n).

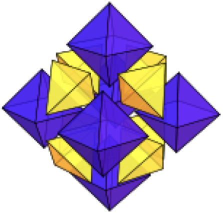
Theorem Every finite locally spherical regular hypertope of euclidean type is a toroidal hypertope (briefly called a toroid).

Example: semi-regular tessellation of E^3

Hypertope of (euclidean) type $\widetilde{B}_3$







Type 0: (red) vertices of the tessellation
Type 1: edges of the tessellation
Type 2: octahedral facets of the tessellation
Type 3: tetrahedral facets of the tessellation

Toroids of rank 3

The regular toroidal maps and hypermaps, that is rank 3 toroids, had been classified in the following families:

$$\{4,4\}_{(s,0)}, \{4,4\}_{(s,s)}, \text{where } s \geq 2$$

$$\{3,6\}_{(s,0)} \text{ where } s \geq 2\ , \{3,6\}_{(s,s)}, \text{ where } s \geq 1$$

$$(3,3,3)_{(s,0)} \text{ where } s \geq 2,\ (3,3,3)_{(s,s)} \text{ where } s \geq 2$$

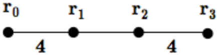
(The vectors in the subscripts determine in each case the translation subgroup used, and the restriction on s guaranties that the hypertopes are large enough so that they do not degenerate.)

Note: Chiral maps and hypermaps have also been classified.

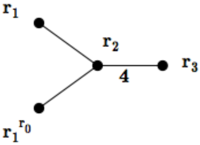
Toroids of rank 4

There is only one affine Coxeter group with linear diagram in rank 4 and two with non-linear diagrams.

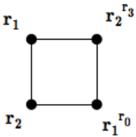
Both groups with non-linear diagrams are subgroups in the group with linear diagram:



$\widetilde{C}_3$



$\widetilde{B}_3$



$\widetilde{A}_3$

Cubic toroids

Let Λ^n be the group of all translational symmetries of E^n (i.e. the lattice $\mathbb{Z}^n$) and $\Lambda^n_{\mathbf{s}}$ the sub-lattice generated by $\mathbf{s} := (s^k, 0^{n-k}),\ k \in \{1, \ldots, n\}$.

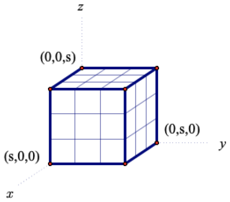
Theorem (McMullen & Schulte 2002) Each regular rank $n+1$ toroid of type $\widetilde{C}_n$, also known as a cubic $(n+1)$-toroid (corresponding to a regular tessellation of n-torus by n-cubes), belong to one of the following three infinite families $\{\widetilde{C}_n\}_{\mathbf{s}} = \{4, 3^{n-2}, 4\}_{\mathbf{s}}$ where

$$\mathbf{s} = (s^k, 0^{n-k}) \textit{ with } s \geq 2 \text{ and } k = 1, 2, \text{or } n,$$

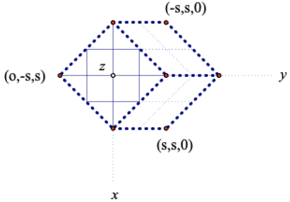
and where the quotient of $\{4, 3^{n-2}, 4\}$ by lattice $\Lambda^n_{\mathbf{s}}$ is denoted simply by $\{4, 3^{n-2}, 4\}_{\mathbf{s}}$.

Families of cubic 4-toroids

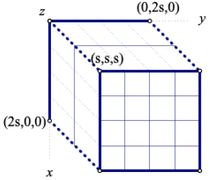
$\{4,3,4\}_{(s,0,0)}$



$\{4,3,4\}_{(s,s,0)}$



$\{4,3,4\}_{(s,s,s)}$



Proper toroidal hypertopes in higher ranks

Theorem (Ens 2016)

- Finite toroidal hypertopes of rank 4 and type $\widetilde{B}_3$, belong to one of the following three infinite families of type $\{\widetilde{B}_3\}_{\mathbf{s}}$ where

$$\mathbf{s} = (2s, 0, 0) \text{ or } (s, s, 0) \text{ with } s \geq 2, \text{ or } (2s, 2s, 2s) \text{ with } s \geq 1.$$

- Finite toroidal hypertopes of rank 4 and type $\widetilde{A}_3$, belong to one of the following three infinite families of type $\{\widetilde{A}_3\}_{\mathbf{s}} = (3, 3, 3, 3)_{\mathbf{s}}$ where

$$\mathbf{s} = (2s, 0, 0) \text{ or } (s, s, 0) \text{ with } s \geq 2, \text{ or } (2s, 2s, 2s) \text{ with } s \geq 1.$$

In both cases the quotient of the universal hypertope of type $\mathcal{C}$ by lattice $\Lambda_{\mathbf{s}}^n$ is denoted simply by $\mathcal{C}_{\mathbf{s}}$.

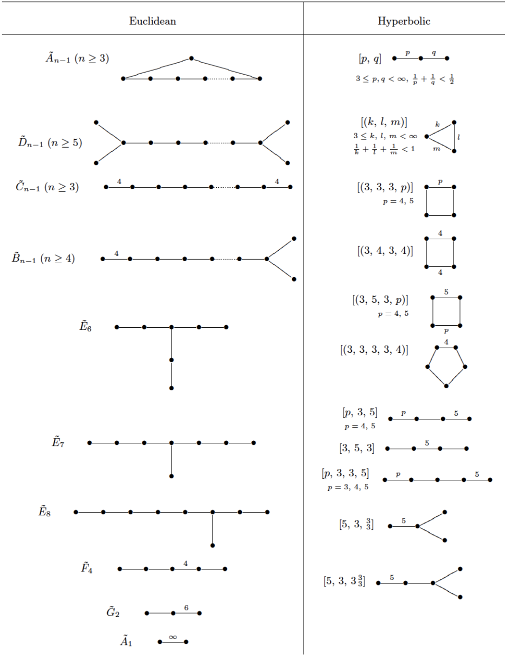
Note: Classification of toroidal hypertopes in higher ranks is partially completed (work in progress with Leemans and Schulte).

Locally spherical hypertopes of hyperbolic type

A regular locally spherical hypertope is of hyperbolic type if it is a hypertope whose Coxeter diagram is the same as the Coxeter diagram of a compact hyperbolic Coxeter group (that is, a group generated by hyperbolic reflexions with compact fundamental domain).

$\Rightarrow$ Regular locally spherical hypertopes are either of spherical, euclidean, or hyperbolic type.

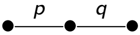
Compact hyperbolic Coxeter groups exist only in ranks 3, 4, and 5.


Euclidean
Hyperbolic
$\tilde{A}_{n-1}$ $(n \geq 3)$
$\tilde{D}_{n-1}$ $(n \geq 5)$
$\tilde{C}_{n-1}$ $(n \geq 3)$
4
4
$\tilde{B}_{n-1}$ $(n \geq 4)$
4
$\tilde{E}_6$
$\tilde{E}_7$
$\tilde{E}_8$
$\tilde{F}_4$
4
$\tilde{G}_2$
6
$\tilde{A}_1$
∞
$[p,\, q]$
p
q
$3 \leq p, q < \infty,\ \frac{1}{p} + \frac{1}{q} < \frac{1}{2}$
$[(k,\, l,\, m)]$
$3 \leq k,\, l,\, m < \infty$
$\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$
k
l
m
$[(3,\, 3,\, 3,\, p)]$
$p = 4,\, 5$
p
$[(3,\, 4,\, 3,\, 4)]$
4
4
$[(3,\, 5,\, 3,\, p)]$
$p = 4,\, 5$
5
p
$[(3,\, 3,\, 3,\, 3,\, 4)]$
4
$[p,\, 3,\, 5]$
$p = 4,\, 5$
p
5
$[3,\, 5,\, 3]$
5
$[p,\, 3,\, 3,\, 5]$
$p = 3,\, 4,\, 5$
p
5
$[5,\, 3,\, 3^{3}_{3}]$
5
$[5,\, 3,\, 3\, 3^{3}_{3}]$
5

Examples of hyperbolic type

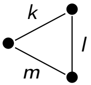
Rank 3 regular hypertopes of hyperbolic type are

- non-degenerate maps with Coxeter diagrams are



with $3 \leq p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, or

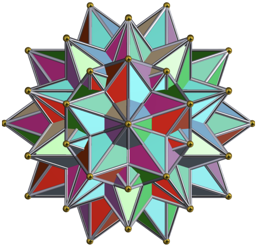
- non-degenerate hypermaps with Coxeter diagrams



with $3 \leq k, l, m < \infty$ and $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$.

The classical regular star-polytope $\{\frac{5}{2}, 3, 5\}$ in E^4, obtained by a sequence of several stellations of 120-cell $\{5, 3, 3\}$, is a finite locally spherical hypertope of hyperbolic type $\{5, 3, 5\}$.

The projection of great stellated 120-cell on the Coxeter plane:



The automorphism group of the polytope is $H_4 = [5, 3, 3] = [5, 3, 5|3]$ (of order 14400) can be obtained from $[5, 3, 5]$ by imposing the extra relation $(\rho_0\rho_1\rho_2\rho_3\rho_2\rho_1)^3 = id$.

THANK YOU