On the cop number of algebraic graphs

AGTIW

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Introduction and Background

Cops and Robbers

Let Γ be a graph. Consider two sets of players, the cops and the robber. First, the cops each choose a vertex in the given graph. Then, the robber chooses a vertex. The game begins and the cops and the robber alternate in taking turns. A cop turn consists of each cop either staying put or moving to an adjacent vertex. A robber turn consists of the robber either staying put or moving to an adjacent vertex.

The goal of the cops is for one cop to occupy the same vertex as the robber, which is called "capturing" the robber. The goal of the robber is to evade capture forever.

Cop Number

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Remarks:

1. It is a two player strategic game played on a graph. We shall consider the game on finite graphs.
2. The game in this form was introduced by Quilliot in 1978 and independently by Nowakowski and Winkler in 1983.
3. A reference to the game of cops and robbers exists in a book of puzzles by Dudeney from as early as 1917 (cf. Bonato).
4. The game is played with perfect information i.e., everyone knows the positions of the others.

Notations I

Definition (Graph)
A graph $\Gamma = (V, E)$ is a tuple where V is an arbitrary set and $E \subseteq V \times V$ (E can be a multi-set), called respectively the set of vertices of Γ and the set of edges of Γ.

Γ is said to be finite if $|V| < +\infty$ and undirected if $(u, v) = (v, u), \forall (u, v) \in E$. We will consider finite, undirected graphs.

Works on the cop-number

Cop-win graphs and cop number

Studied by numerous authors including Aigner–Fromme, Andreae, Hamidoune, Maamoun–Meyniel, Nowakowski–Winkler, Quilliot and others since the 1980s.

1. Study of cop-win graphs where a cop-win graph is an undirected graph in which the cop can always win.
2. Study of the cop number.

Meyniel's conjecture and weak Meyniel's conjecture

Conjecture (Meyniel 1987)
The cop number of a connected graph on n vertices is $O(\sqrt{n})$.

It is also best possible as there exist graphs for which the above bound is attained.

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Conjecture (Weak Meyniel)
For a fixed constant $c > 0$, the cop number of a graph on n vertices is $O(n^{1-c})$.

Not just Meyniel's conjecture, but even the weak Meyniel's conjecture is open in general.

Developments around Meyniel's conjecture

1. Hamidoune, d-regular Cayley graph of an abelian group, cop-number is at most $\frac{3d}{4}$.
2. Frankl, d-regular Cayley graph of an abelian group, cop-number is at most $\frac{d+1}{2}$, for full Cayley graphs it is at most d.
3. Chiniforooshan, $c(\Gamma) = O\left(\frac{n}{\log n}\right)$ (general connected graphs)
4. Lu–Peng, Frieze–Krivelevich–Loh, Scott–Sudakov, $c(\Gamma) = O\left(\frac{n}{2^{1-o(1)\sqrt{\log_2 n}}}\right)$ (general connected graphs)
5. Bollobas–Kun–Leader, if $c(\Gamma) = O(\sqrt{n} logn)$ (a.a.s. with some condition on p for binomial random graphs)

Developments around Meyniel's conjecture II

1. Pralat–Wormald, $c(\Gamma) = O(\sqrt{n})$ (a.a.s. with some condition on p for binomial random graphs and for random d-regular graphs)
2. Bradshaw, Meyniel's conjecture for undirected abelian Cayley graphs
3. Bradshaw–Hosseini–Turcotte, Meyniel's conjecture for abelian Cayley digraphs
4. Bradshaw–Hosseini–Mohar–Stacho, Meyniel's conjecture holds for expander graph families of bounded degree.
5. Hosseini–Mohar–de la Maza, Meyniel's conjecture for graphs of bounded degree implies weak Meyniel's conjecture for all graphs.

Cop number of algebraic graphs

Algebraic graphs

Let G be a finite group and S be a non-empty subset of G. The Cayley graph $C(G, S)$ (also sometimes known as the *difference Cayley graph*) is the graph having the elements of G as vertices and there is an edge from a vertex u to a vertex v if $v = us$ for some $s \in S$.

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Let σ be a group automorphism of G. The twisted Cayley graph $C(G, S)^\sigma$ has G as its set of vertices, and for $x, y \in G$, there is an edge from x to y if $y = \sigma(xs)$ for some $s \in S$.

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Remarks on algebraic graphs

1. The four classes of graphs described above can be structurally quite different.
2. The twisted Cayley graphs are related to the generalised Cayley graphs first introduced by by Marušič–Scapellato–Zagaglia Salvi in 1991. The twisted Cayley graphs with respect to involutions and that do not contain any loop are precisely the generalised Cayley graphs.
3. Cayley graphs are vertex transitive, but the other three classes are not in general. This lack of symmetry sometimes makes questions difficult to study. Vertex-connectivity of abelian Cayley sum graphs is a non-trivial problem and was treated by Grynkiewicz–Lev–Serra in 2009, existence of vertex transitive non Cayley but generalised Cayley graphs was shown in 2016 by Hujdurović–Kutnar–Marušič etc.

Results obtained

Theorem

1. *Suppose the Cayley sum graph* $C_\Sigma(G,S)$ *is undirected and connected. Assume that* S *is symmetric. Then the cop number of* $C_\Sigma(G,S)$ *is at most* $|S|$.
2. *Let* σ *be an automorphism of* G *of order two. Suppose the twisted Cayley graph* $C(G,S)^\sigma$ *is undirected and connected. Assume that* S *is closed under conjugation by the elements of* G *and is symmetric. Then the cop number of* $C(G,S)^\sigma$ *is at most* $|S|$.
3. *Let* σ *be an automorphism of* G *of order two. Suppose the twisted Cayley sum graph* $C_\Sigma(G,S)^\sigma$ *is undirected and connected and* S *is symmetric. Then the cop number of* $C_\Sigma(G,S)^\sigma$ *is at most* $|S|$.

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3. *Let σ be an automorphism of G of order two. Suppose the twisted Cayley sum graph $C_\Sigma(G,S)^\sigma$ is undirected and connected and S is symmetric. Then the cop number of $C_\Sigma(G,S)^\sigma$ is at most $|S|$.*

Theorem
The weak Meyniel's conjecture holds for full Cayley graphs, and the algebraic graphs considered above.

$C_\Sigma(G, S)$ - Outline of Proofs

We adopt the convention that during the zeroth round, the cops and the robber occupy their initial positions. For $n \geq 1$, during the n-th round of the cops and robber game on a graph, the robber takes its n-th turn and the cops take their n-th turn.

Let $|S|$ cops occupy the vertex at the identity element of G, and the robber occupy some vertex of $C_\Sigma(G, S)$. At the end of the zeroth round, i.e. after the cops and the robber have occupied their initial positions on $C_\Sigma(G, S)$, we label the cops using the elements of S, and the cop corresponding to an element s of S is denoted by $c(s)$.

The cops will be relabelled at the end of each round.

$C_\Sigma(G, S)$ - Outline of Proofs

Let $n \geqslant 1$. Suppose at the beginning of the n-th round, the robber is at the vertex y and moves to $y^{-1}t$ for some $t \in S$, and a cop C having label $c(s)$ is positioned at the vertex z. Since $C_\Sigma(G, S)$ is connected, S generates G and one can write $z^{-(-1)^{n-1}}y^{(-1)^{n-1}} = w_{z,y,k}s^i$ where i is an integer, $w_{z,y,k}$ denotes the product of (a choice of) k elements of S with k as small as possible. If n is odd, then the cop C

1. moves to $(y^{-1}t^{-1}y)^{-1}z^{-1}$ if $y^{-1}t^{-1}y$ is not a power of s,
2. moves to $s^{-1}z^{-1}$ if $y^{-1}t^{-1}y \in \{s, s^{-1}\}$ and $k = 0$,
3. moves to $(y^{-1}t^{-1}y)^{-1}z^{-1}$ if $y^{-1}t^{-1}y$ is a power of s, the element $y^{-1}t^{-1}y$ does not lie in $\{s, s^{-1}\}$ and $k = 0$,
4. moves to $g_1^{-1}z^{-1}$ if $y^{-1}t^{-1}y$ is a power of s and $k \geq 1$ with $w_{z,y,k} = g_1g_2\cdots g_k$ for some $g_1, g_2, \ldots, g_k \in S$,

and the label of C is changed to $c((y^{-1}t^{-1}y)^{-1}sy^{-1}t^{-1}y)$.

The above moves are possible in $C_\Sigma(G, S)$

$C_\Sigma(G, S)$ - Outline of Proofs

For any $x \in S$, there is an edge from z to any vertex of the form $x^{-1}z^{-1}$ since $x^{-1}z^{-1} = z^{-1}(zx^{-1}z^{-1})$ and S is closed under conjugation and S is symmetric.

If n is even, then the cop C

1. moves to $z^{-1}t$ if t is not a power of s,
2. moves to $z^{-1}s$ if $t \in \{s, s^{-1}\}$ and $k = 0$,
3. moves to $z^{-1}t$ if t is a power of s, the element t does not lie in $\{s, s^{-1}\}$ and $k = 0$,
4. moves to $z^{-1}g_1$ if t is a power of s and $k \geq 1$ with $w_{z,y,k} = g_1g_2 \cdots g_k$ for some $g_1, g_2, \ldots, g_k \in S$,

and the label of C is changed to $c(t^{-1}st)$.

$C_\Sigma(G, S)$ - **Connection, tail, power**

We introduce the notions of

1. Connection of the robber: If the robber moves from y to $y^{-1}t$ at the beginning of the n-th round, then its *connection* is defined as the element $y^{-1}t^{-1}y$ or t of S according as n is odd or even.

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2. The *tail* of a cop: If n is odd, we write $z^{-1}y = w_{z,y,k}s^i$ where i is an integer, $w_{z,y,k}$ denotes the product of (a choice of) k elements of S with k as small as possible, and call k the *tail* of the cop with label $c(s)$ at the beginning of the n-th round. If n is even, we write $zy^{-1} = w_{z,y,k}s^i$ where i is an integer, $w_{z,y,k}$ denotes the product of (a choice of) k elements of S with k as small as possible, and call k the *tail* of the cop C with label $c(s)$ at the beginning of the n-th round.

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3. The *power* of a cop with tail zero: If C has tail zero at the beginning of the n-th round, then the *power* of C is defined as the smallest non-negative integer i such that $z^{-(-1)^{n-1}}y^{(-1)^{n-1}} = s^i$.

$C_{\Sigma}(G, S)$ - Reduction of tail and power

Lemma

Let n be a positive integer, and C be a cop. Then the tail of C at the beginning of the n-th round is greater than or equal to its tail at the beginning of the $(n+1)$-st round. If C has tail zero and power α at the beginning of the n-th round, then the power of C at the beginning of the $(n+1)$-st round is

1. *α if the connection of the robber during the n-th round is equal to the label of C,*
2. *$\alpha - 2$ if the connection of the robber during the n-th round is the inverse of the label of C,*
3. *α if the connection of the robber during the n-th round is a power of the label of C other than the label or its inverse.*

Further, if n is odd and C has tail zero and power 1 at the beginning of the n-th round, then the robber will be captured by C during the n-th round if the connection of the robber during the n-th round is equal to the label of C.

Generalised / Twisted Cayley graphs $C(G,S)^\sigma$

If n is odd, then the cop C

1. moves to $\sigma(zt)$ if t is not a power of s,
2. moves to $\sigma(zs)$ if $t \in \{s, s^{-1}\}$ and $k = 0$,
3. moves to $\sigma(zt)$ if t is a power of s, the element t does not lie in $\{s, s^{-1}\}$ and $k = 0$,
4. moves to $\sigma(zg_1)$ if t is a power of s and $k \geq 1$ with $w_{z,y,k} = g_1 g_2 \cdots g_k$ for some $g_1, g_2, \ldots, g_k \in S$,

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and the label of C is changed to $c(t^{-1}st)$. If n is even, then the cop C

1. moves to $\sigma(zt)$ if $\sigma(t)$ is not a power of s,
2. moves to $\sigma(z)s$ if $\sigma(t) \in \{s, s^{-1}\}$ and $k = 0$,
3. moves to $\sigma(zt)$ if $\sigma(t)$ is a power of s, the element $\sigma(t)$ does not lie in $\{s, s^{-1}\}$ and $k = 0$,
4. moves to $\sigma(z)g_1$ if $\sigma(t)$ is a power of s and $k \geq 1$ with $w_{z,y,k} = g_1 g_2 \cdots g_k$ for some $g_1, g_2, \ldots, g_k \in S$,

and the label of C is changed to $c(\sigma(t)^{-1} s \sigma(t))$.

Generalised / Twisted Cayley sum graphs $C_\Sigma(G,S)^\sigma$

If n is odd, then the cop C

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3. moves to $\sigma((y^{-1}t^{-1}y)^{-1}z^{-1})$ if $y^{-1}t^{-1}y$ is a power of s, the element $y^{-1}t^{-1}y$ does not lie in $\{s, s^{-1}\}$ and $k=0$,
4. moves to $\sigma(g_1^{-1}z^{-1})$ if $y^{-1}t^{-1}y$ is a power of s and $k \geq 1$ with $w_{z,y,k} = g_1 g_2 \cdots g_k$ for some $g_1, g_2, \ldots, g_k \in S$,

and label of C is changed to $c((y^{-1}t^{-1}y)^{-1}sy^{-1}t^{-1}y)$.

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3. moves to $\sigma(z^{-1}t)$ if $\sigma(t)$ is a power of s, the element $\sigma(t)$ does not lie in $\{s, s^{-1}\}$ and $k = 0$,
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and the label of C is changed to $c(\sigma(t)^{-1}s\sigma(t))$.

Weak Meyniel's conjecture

Theorem

1. *The cop number of any undirected, connected full Cayley graph $C(G,S)$ of a group G of order n is at most $2\sqrt{n}\log n$.*
2. *The cop number of any undirected, connected Cayley sum graph $C_\Sigma(G,S)$ of a group G of order n is at most $2\sqrt{n}\log n$, where S is a symmetric subset of G.*
3. *The cop number of any undirected, connected generalised Cayley graph $C(G,S)^\sigma$ of a group G of order n is at most $2\sqrt{n}\log n$, where S is a symmetric subset of G, closed under conjugation by the elements of G.*
4. *The cop number of any undirected, connected twisted Cayley sum graph $C_\Sigma(G,S)^\sigma$ of a group G of order n is at most $2\sqrt{n}\log n$, where S is a symmetric subset of G and σ is of order 2.*

Consequently, for any real number $c < \frac{1}{2}$, the above cop numbers are $O(n^{1-c})$.

Proof

If S is a subset of G satisfying $|S| \geq \sqrt{n}\log n$, then by a result of Bollobas–Janson–Riordan, there exists a subset T of G of order at most $\sqrt{n}(1+\frac{1}{\log n}) \leq 2\sqrt{n}$ such that $TS = G$, i.e., G can be covered by at most $2\sqrt{n}$ left translates of S.

By employing one cop at each of the vertices of $C(G,S)$ (respectively $C_\Sigma(G,S)$, $C(G,S)^\sigma$,$C_\Sigma(G,S)^\sigma$) corresponding to the elements of T (respectively, T^{-1}, $\sigma(T)$, $\sigma(T^{-1})$) when $|S| \geq \sqrt{n}\log n$, it follows that the robber will be captured in $C(G,S)$ (respectively $C_\Sigma(G,S)$, $C(G,S)^\sigma$,$C_\Sigma(G,S)^\sigma$).

If $|S| \leqslant \sqrt{n}\log n$, then apply the previous.

Further directions

Open questions

1. Bounds for Cayley graphs of nilpotent groups?
2. Meyniel's conjecture for other classes of non-abelian Cayley graphs / algebraic graphs?
3. What about finite simple groups? Can growth results in finite simple groups be used to show the conjecture for algebraic graphs based on them?

THANK YOU