Constructing cospectral hypergraphs

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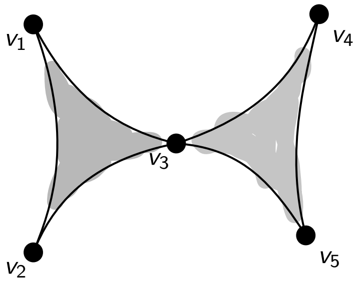
Outline

Introduction

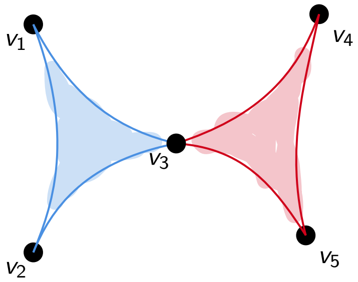


A **hypergraph** is a generalization of a graph in which an edge may have any number of vertices.

k-**uniform hypergraph**: every edge holds k vertices.

Example. 3-uniform hypergraph on 5 vertices, the edge set is $E = \{v_1v_2v_3, v_3v_4v_5\}$.

2-uniform hypergraph = (ordinary) graph.



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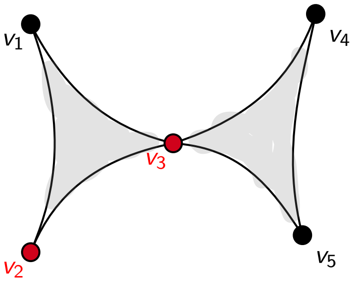
Hypergraphs



The **neighborhood** $\Gamma(U)$ of U is a set of all vertices x such that $x \cup U$ is an edge.

Example. The neighborhood of $\{v_2, v_3\}$ is $\Gamma(v_2, v_3) = v_1$.

Spectral graph theory ($\sim$1950s) aims to obtain structural information about graphs from their spectra.

Many known results for *graphs*, e.g:

- Regularity and bipartiteness can be determined from the spectrum
- Eigenvalue bounds on the chromatic number
- Almost all trees or strongly regular graphs are cospectral
- Haemers's conjecture: almost all graphs are determined by their spectrum
- ...

Spectral hypergraph theory ($\sim$1990s) aims to obtain structural information about hypergraphs from their spectra.

Some results in spectral hypergraph theory include (see *e.g.* Cooper, Dutle, 2012):

- Regularity and bipartiteness can be determined from hypergraph spectrum
- Eigenvalue bounds on the chromatic number
- ...

Very few results are known for spectral characterizations of hypergraphs.

Bu, Zhou, and Wei (2014) showed that the following families of hypergraphs are determined by their spectra:

- complete k-uniform hypergraphs and their complements,
- complete k-uniform hypergraphs without one edge,
- subhypergraphs of complete $(n-1)$-uniform hypergraphs.

Goal

Two hypergraphs are **cospectral** if they have the same spectrum (eigenvalues).

Studying *cospectral* graphs (hypergraphs) helps us reveal which structural properties cannot be deduced from the spectra.

Methods to construct cospectral *graphs* include:

- GM-switching (Godsil, McKay, 1982),
- *WQH-switching* (Wang, Qiu, Hu, 2019),
- ...

Our goal is to obtain new methods to construct *cospectral hypergraphs*.

The spectrum of a hypergraph can be calculated in different ways based on how the hypergraph is represented:

1. adjacency tensor, or hypermatrix (Cooper, Dutle, 2012),
2. integer matrix with number of common edges as its entries (Feng, Li, 1996),
3. other: $\{0,1\}$-adjacency matrix, distance matrix, ...

We focus on the *first two representations*.

The GM-switching procedure has been generalized to hypergraphs:

	GM-switching	WQH-switching
	Godsil, McKay (1982)	Wang, Qiu, Hu (2019)
	↓	↓
adjacency tensor:	Bu, Zhou, Wei (2014)	?
matrix representation:	Sarkar, Banerjee (2020)	?

Constructing cospectral hypergraphs with respect to **tensors**

Adjacency tensor of a hypergraph

For a *k-uniform hypergraph* on n vertices we can define the **adjacency tensor** $\mathcal{A} = (a_{i_1 \cdots i_k})$ of order k dimension n:

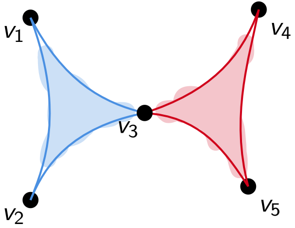
$$a_{i_1 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \{i_1, \ldots, i_k\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

NB! The adjacency tensor can be only defined for *uniform* hypergraphs.

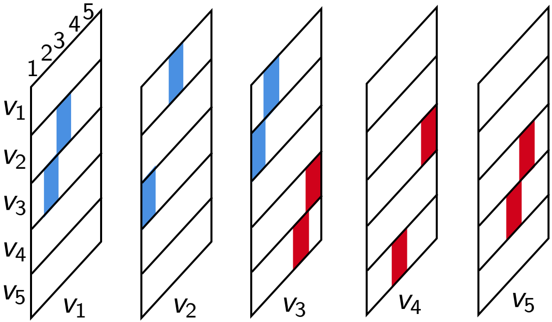
Computing eigenvalues of a tensor is NP-hard (Hillar, Lim, 2013).

Adjacency tensor of a hypergraph

$$a_{i_1 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \{i_1, \ldots, i_k\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$



Example. Tensor of dimension 5 order 3:



Eigenvalues of a tensor

What are the eigenvalues of a tensor?

For a MATRIX A dimension n:

- λ is an *eigenvalue* if $Ax = \lambda x$ for some vector $x \neq 0$ and $x^\top x = 1$, or equivalently,
- λ is root of characteristic polynomial $\varphi_A(\lambda) = \det(\lambda I_n - A)$.

What are the eigenvalues of a tensor?

For a TENSOR $\mathcal{A}$ of order k dimension n there are *two* definitions (Qi, 2005, and Lim, 2005):

- λ is an **eigenvalue** if it is a root of characteristic polynomial $\Phi_{\mathcal{A}}(\lambda) = \det(\lambda \mathcal{I}_n - \mathcal{A})$.
- λ is an **E-eigenvalue** if $\mathcal{A}x = \lambda x$ for some $x \neq 0$ and $x^\top x = 1$. The tensor product in $\mathcal{A}x$ is defined similarly to the usual matrix product (Shao, 2013).

Two hypergraphs are **cospectral (E-cospectral)** if they have the same eigenvalues (E-eigenvalues).

In this talk we will construct hypergraphs that are *both* cospectral and E-cospectral.

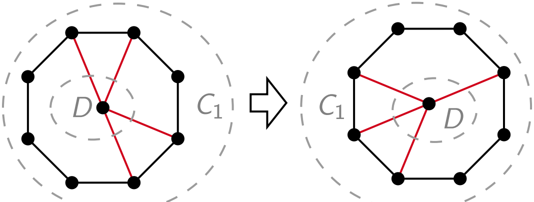
GM-switching (ordinary graphs)

Theorem (Godsil, McKay, 1982)

Let G be a graph whose vertex set admits a partition $C_1 \cup C_2 \cup \cdots \cup C_m \cup D$ such that:

1. for any $i \leq m$ each vertex in D has either 0, $\frac{1}{2}|C_i|$, or $|C_i|$ neighbors in C_i.
2. (Equitable partition) for all $i, j \leq m$ every vertex in C_i has the same number of neighbors in C_j.

To construct graph H, for any $v \in D$ that has $\frac{1}{2}|C_i|$ neighbors in C_i switch the adjacency of $\{u, v\}$ for all $u \in C_i$. Then H is a cospectral graph with G.



WQH-SWITCHING (ORDINARY GRAPHS)

THEOREM (WANG, QIU, HU, 2019)

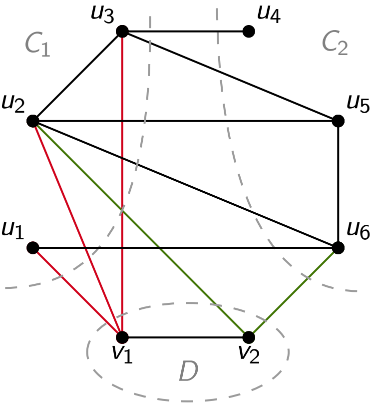
Let G be a graph whose vertex set admits a partition $C_1 \cup C_2 \cup D$ such that:

1. $|C_1| = |C_2|$.
2. There exists a constant c such that for any $v \in C_i$ we have $|\Gamma(v) \cap C_j| - |\Gamma(v) \cap C_i| = c$, where $\{i, j\} = \{1, 2\}$.
3. For any vertex $v \in D$ we have either $\Gamma(v) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$ or $|\Gamma(v) \cap C_1| = |\Gamma(v) \cap C_2|$.

To construct a graph H, for any $v \in D$ such that $\Gamma(v) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$ switch the adjacency of $\{u, v\}$ for any $u \in C_1 \cup C_2$. Then H is a cospectral graph with G.

A generalized version of this switching for partition of vertices into $2m + 1$ subsets $C_1 \cup \cdots \cup C_{2m} \cup D$ was described by Qiu, Ju, Wang (2020).

WQH-SWITCHING (EXAMPLE)



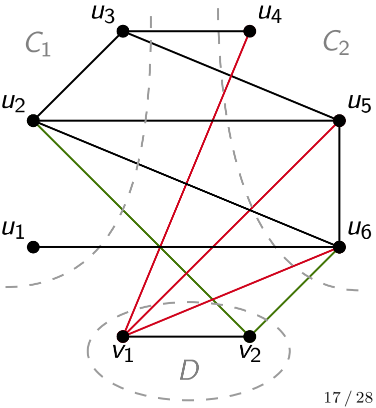
$\Gamma(v_1) \cap (C_1 \cup C_2) = C_1$

$|\Gamma(v_2) \cap C_1| = |\Gamma(v_2) \cap C_2|$

$|\Gamma(u_2) \cap C_2| - |\Gamma(u_2) \cap C_2| = 2 - 1 = 1$

$|\Gamma(u_1) \cap C_2| - |\Gamma(u_1) \cap C_2| = 1 - 0 = 1$

$\dots$



Switching edges: $v_1 u_1$, $v_1 u_2$, $v_1 u_3$ ⇨ $v_1 u_4$, $v_1 u_5$, $v_1 u_6$

New switching for hypergraphs (tensors)

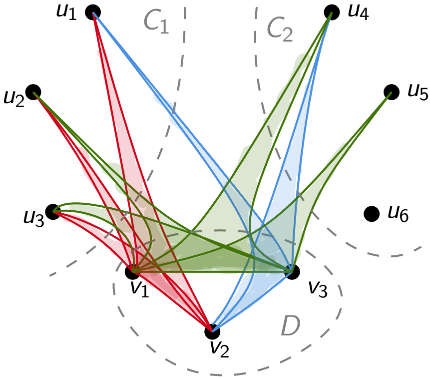
Theorem 1 (Abiad, K)

Let G be a k-uniform hypergraph whose vertex set admits a partition $C_1 \cup C_2 \cup D$, and such that:

1. $|C_1| = |C_2|$.
2. Any edge has at least $k-1$ vertices in D.
3. For any $k-1$ distinct vertices $u_2, \ldots, u_k$ from D, we have $\Gamma(u_2, \ldots, u_k) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$ or $|\Gamma(u_2, \ldots, u_k) \cap C_1| = |\Gamma(u_2, \ldots, u_k) \cap C_2|$.

To construct a hypergraph H, for any $\{u_2, \ldots, u_k\} \subseteq D$ such that its neighbors in $C_1 \cup C_2$ are all in C_1 (or C_2), switch the adjacency of $\{u_1, \ldots, u_k\}$ for all $u_1 \in C_1 \cup C_2$. Then H is a k-uniform cospectral (E-cospectral) hypergraph with G.

Condition 2 implies that in $C_1 \cup C_2$ no two vertices are adjacent.

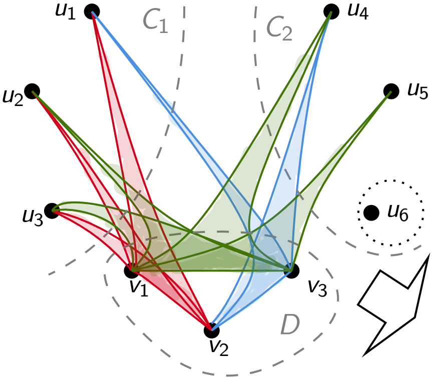


Edges:

		$v_1 v_3 u_2$
	$v_1 v_2 u_1$	$v_1 v_3 u_3$
$v_2 v_3 u_1$	$v_1 v_2 u_2$	$v_1 v_3 u_4$
$v_2 v_3 u_4$	$v_1 v_2 u_3$	$v_1 v_3 u_5$

- $|C_1| = |C_2| = 3.$
- Every edge has 2 vertices in D.
- $\Gamma(v_1, v_2) = C_1$ (no neighbors in C_2);
- v_2, v_3 have one neighbor in each C_1 and C_2;
- v_1, v_3 have 2 neighbors in each C_1 and C_2.

New switching for hypergraphs (tensors)



Common edges:

$v_2 v_3 u_1$

$v_2 v_3 u_4$

$v_1 v_3 u_2$

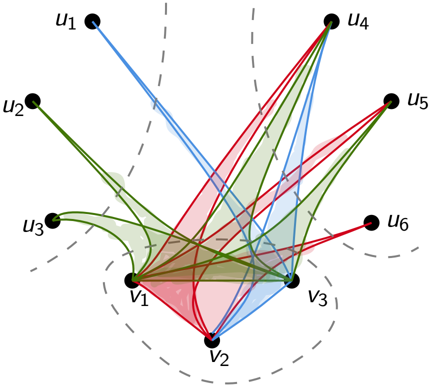
$v_1 v_3 u_3$

$v_1 v_3 u_4$

$v_1 v_3 u_5$

Switching edges:

$v_1 v_2 u_1$	⇒	$v_1 v_2 u_4$
$v_1 v_2 u_2$		$v_1 v_2 u_5$
$v_1 v_2 u_3$		$v_1 v_2 u_6$



New switching for hypergraphs (tensors)

Theorem 1+ (Abiad, K)

Let G be a k-uniform hypergraph whose vertex set admits a partition $C_1 \cup C_2 \cup \cdots \cup C_{2m} \cup D$ for some $m \geq 1$, and such that:

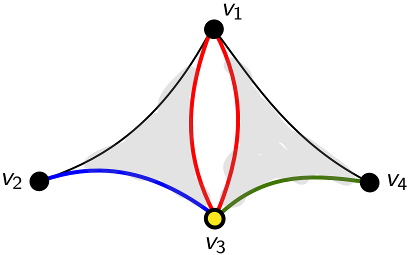
1. $|C_i| = |C_{i+1}|$ for all odd $i < 2m$.
2. Any edge has at least $k-1$ vertices in D.
3. For any odd $i < 2m$ and $k-1$ distinct vertices $u_2, \ldots, u_k$ from D, we have $\Gamma(u_2, \ldots, u_k) \cap (C_i \cup C_{i+1}) \in \{C_i, C_{i+1}\}$ or $|\Gamma(u_2, \ldots, u_k) \cap C_i| = |\Gamma(u_2, \ldots, u_k) \cap C_{i+1}|$.

To construct a hypergraph H, for any $\{u_2, \ldots, u_k\} \subseteq D$ such that its neighbors in $C_i \cup C_{i+1}$ are all in C_i (or C_{i+1}), switch the adjacency of $\{u_1, \ldots, u_k\}$ for all $u_1 \in C_i \cup C_{i+1}$. Then H is a k-uniform cospectral (E-cospectral) hypergraph with G.

Constructing cospectral hypergraphs with respect to **matrices**

First proposed by Feng and Li (1996), a similar definition used by Sarkar and Banerjee (2020) in GM-switching for hypergraphs.

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$



The entry a_{ij} of $A = (a_{ij})$ is the number of edges that contain both v_i and v_j.

It is not as immediately clear what the edges are just from the matrix alone as it is when using the tensor definition, but the computation of eigenvalues can be done in polynomial time.

New switching for hypergraphs (matrices)

Theorem 2 (Abiad, K)

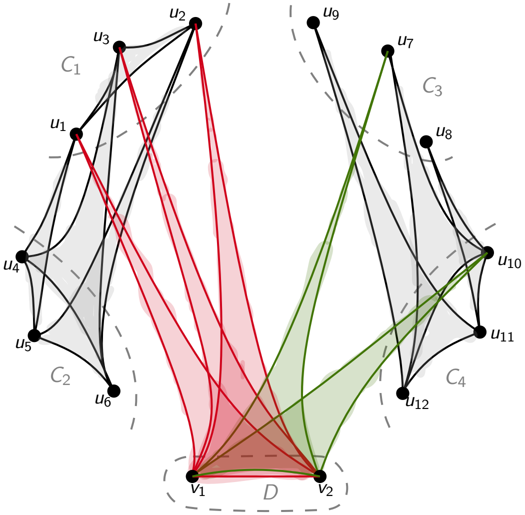
Let G be a k-uniform hypergraph whose vertex set admits a partition $C_1 \cup C_2 \cup \cdots \cup C_{2m} \cup D$ for some $m \geq 1$, and such that:

1. $|C_i| = t$ for all i and some t, while $|D| = k - 1$.
2. Any edge of G has 0 or $k-1$ vertices in D.
3. For any odd $i < 2m$, we have either $\Gamma(D) \cap (C_i \cup C_{i+1}) = C_i$ or $|\Gamma(D) \cap C_i| = |\Gamma(D) \cap C_{i+1}|$.
4. For the adjacency matrix A and each $i, j \leq 2m$ there exists α_{ij} such that

$$\sum_{u \in C_i} A_{uv} = \sum_{u \in C_i} A_{vu} = \sum_{u \in C_j} A_{uw} = \sum_{u \in C_j} A_{wu} = \alpha_{ij} \text{ for } v \in C_j,\ w \in C_i.$$

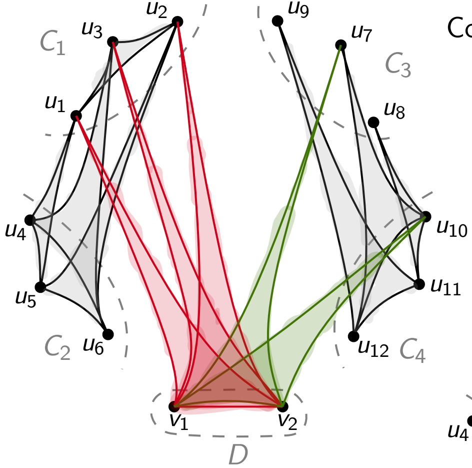
To construct a hypergraph H, remove all edges (v, D) such that $v \in C_i$ and $\Gamma(D) \supseteq C_i$ and add edges (u, D) with $u \in C_{i+1}$, for all odd $i < 2m$. Then H is cospectral to G with respect to matrix representation.

New switching for hypergraphs (matrices)



- $|C_i| = 3$ for $i = 1, 2, 3, 4$.
- $|D| = 2$ in 3-uniform hypergraph.
- Every edge has 0 or 2 vertices in D.
- v_1, v_2 are adjacent to all of C_1 and none of C_2.
- v_1, v_2 are adjacent to one vertex in both C_3 or C_4.
- number of neighbors in C_j is the same for all $v \in C_i$, $i, j = 1, 2, 3, 4$.

New switching for hypergraphs (matrices)



Common edges:

$u_1 u_2 u_3$ $u_7 u_{10} u_{12}$ $u_7 v_1 v_2$

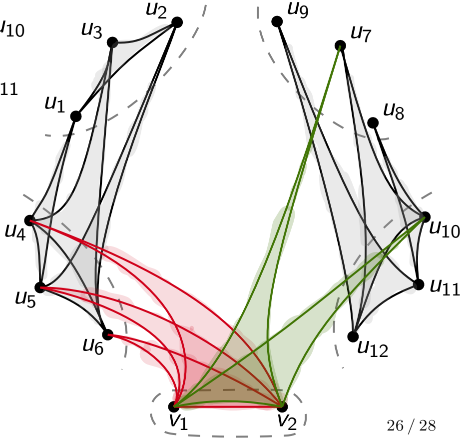
$u_1 u_4 u_5$ $u_8 u_{10} u_{11}$ $u_{10} v_1 v_2$

$u_2 u_5 u_6$ $u_9 u_{11} u_{12}$

$u_3 u_4 u_6$

Switching edges:

$v_1 v_2 u_1$	⇨	$v_1 v_2 u_4$
$v_1 v_2 u_2$		$v_1 v_2 u_5$
$v_1 v_2 u_3$		$v_1 v_2 u_6$



Conclusion and future research

Conclusion and future research

	GM-switching	WQH-switching	other methods?
	Godsil, McKay (1982)	Wang, Qiu, Hu (2019)	
	$\downarrow$	$\downarrow$	$\downarrow$
tensor:	Bu, Zhou, Wei (2014)	Theorem 1	?
matrix:	Sarkar, Banerjee (2020)	Theorem 2	?

- What other results of spectral graph theory admit an extension to hypergraphs?
- Tools that could be used to get new results on spectral characterization of hypergraphs?

Thank you for your attention!

Preprint on arXiv:

Abiad, A., Khramova, A.P.

Constructing cospectral hypergraphs.

`https://arxiv.org/abs/2211.06087`