Construction of graphs from groups

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Outline of the talk:

1. Basic definitions and preliminaries
2. Tactical decomposition and orbit matrices
3. The method
4. Example - Deza graph
5. Example - SRG(81,30,12,9)

An **incidence structure** is an ordered triple $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$.

The elements of the set $\mathcal{P}$ are called **points**, the elements of the set $\mathcal{B}$ are called **blocks** and $\mathcal{I}$ is called an **incidence relation**.

An isomorphism from one incidence structure to another is a bijective mapping of points to points and blocks to blocks which preserves incidence. An isomorphism from an incidence structure $\mathcal{D}$ onto itself is called an **automorphism** of $\mathcal{D}$.

The set of all automorphisms forms a group called the **full automorphism group** of $\mathcal{D}$ and is denoted by $Aut(\mathcal{D})$.

A t-(v,k,λ) **design** is a finite incidence structure $\mathcal{D}=(\mathcal{P},\mathcal{B},\mathcal{I})$ satisfying the following requirements:

1. $|\mathcal{P}|=v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

Every element of $\mathcal{P}$ is incident with exactly $r=\frac{\lambda(v-1)}{k-1}$ elements of $\mathcal{B}$.

The number of blocks is denoted by b.

If $b=v$ (or equivalently $k=r$) then the design is called **symmetric**.

- A 2-(v, k, λ) design is called a block design.
- If $\mathcal{D}$ is a t-design, then it is also a s-design, for $1 \leq s \leq t-1$.
- An **incidence matrix** of a design $\mathcal{D}$ is a matrix $A = [a_{ij}]$ where $a_{ij} = 1$ if jth point is incident with the ith block and $a_{ij} = 0$ otherwise.

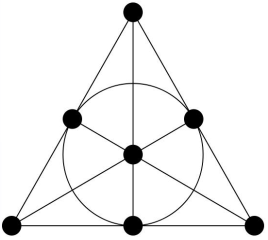


Figure: 2-$(7, 3, 1)$ design

- P. Dembowski, Verallgemeinerungen von Transitivitatsklassen endlicher projektiver Ebenen, Math. Z. 69, 1958.
- H. Beker, C. Mitchell, F. Piper, Tactical decompositions of designs, Aequationes mathematicae, vol. 25, pp 123–152, 1982.

Let A be the incidence matrix of an **incidence structure** $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$.

A **decomposition** of A is any partition $B_1, \ldots, B_m$ of the rows of A (blocks of $\mathcal{D}$) and a partition $P_1, \ldots, P_n$ of the columns of A (points of $\mathcal{D}$).

For $i \leq m$, $j \leq n$ define

$$\alpha_{ij} = |\{P \in P_j |\ P\mathcal{I}x\}|, \text{ for } x \in B_i \text{ arbitrarily chosen,}$$
$$\beta_{ij} = |\{x \in B_i |\ P\mathcal{I}x\}|, \text{ for } P \in P_j \text{ arbitrarily chosen.}$$

We say that a decomposition is **tactical** if the α_{ij} and β_{ij} are well defined (independent from the choice of $x \in B_i$ and $P \in P_j$, respectively).

There are two trivial tactical decompositions of 1-designs:

Decomposition for which $m = n = 1$ (the decomposition contained of just one class of points and blocks). The number of blocks containing a point is constant, and the number of points contained in a block is constant.

Decomposition for which $n = v, m = b$ i.e. the decomposition contained of classes of size 1.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an **incidence structure** and $G \leq Aut(\mathcal{D})$.

We denote the G-orbits of points by $\mathcal{P}_1, \ldots, \mathcal{P}_n$, G-orbits of blocks by $\mathcal{B}_1, \ldots, \mathcal{B}_m$, and put $|\mathcal{P}_r| = \omega_r$, $|\mathcal{B}_i| = \Omega_i$, $1 \leq r \leq n$, $1 \leq i \leq m$.

Proposition

The **group action** of G induces a **tactical decomposition of** $\mathcal{D}$.

Sketch of the proof:
Let $X, Y \in \mathcal{P}_j, X \neq Y$, and let $B_1, \ldots, B_k \in \mathcal{B}_i$ are incident with X. Since X and Y are in the same G-orbit, there exists $g \in G$ such that $g.X = Y$. Then $g.B_1, \ldots, g.B_k$ are blocks from $\mathcal{B}_i$ incident with Y.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a 2-(v, k, λ) design and $G \leq Aut(\mathcal{D})$.

Denote by γ_{ij} the number of points of $\mathcal{P}_j$ incident with a representative of the block orbit $\mathcal{B}_i$. For these numbers the following equalities hold:

$$\sum_{j=1}^{n} \gamma_{ij} = k\,, \qquad (1)$$

$$\sum_{i=1}^{m} \frac{\Omega_i}{\omega_j} \gamma_{ij} \gamma_{is} = \lambda \omega_s + \delta_{js} \cdot (r - \lambda)\,. \qquad (2)$$

A $(m \times n)$-matrix $M = (\gamma_{ij})$ with entries satisfying conditions (1) and (2) is called an **orbit matrix for the parameters** $2-(v, k, \lambda)$ and orbit lengths distributions $(\omega_1, \ldots, \omega_n)$, $(\Omega_1, \ldots, \Omega_m)$.

Example

Incidence matrix for the symmetric (7,3,1) design:

$$\left(\begin{array}{c|ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

Corresponding orbit matrix for Z_3:

	1	3	3
1	0	3	0
3	1	1	1
3	0	1	2

Orbit matrices are often used in construction of designs with a presumed automorphism group:

- Z. Janko, T. van Trung, Construction of a new symmetric block design for (78,22,6) with the help of tactical decompositions, J. Combin. Theory A 40, (1985) 451–455.
- D. Crnković, Symmetric (70,24,8) designs having $\mathrm{Frob}_{21} \times Z_2$ as an automorphism group, Glas. Mat. Ser. III 34 (54), No. 2 (1999), 109–121.
- S. D. Stoichev, V. D. Tonchev, Unital designs in planes of order 16, Discrete Appl. Math. 102 (2000), 151–158.
- D. Crnković, A. Švob, New symmetric 2-(176,50,14) designs, Discrete Math. 344 (2021), 112623.

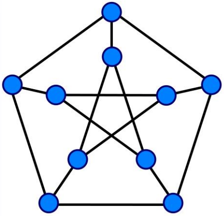
Construction of designs admitting an action of the presumed automorphism group consists of two steps:

1. Construction of orbit matrices for the given automorphism group,
2. Construction of block designs for the obtained orbit matrices.

Strongly regular graph

A graph is **regular** if all the vertices have the same degree. A regular graph is **strongly regular** of type (v, k, λ, μ) if it has v vertices, valency k, and if any two adjacent vertices are together adjacent to λ vertices, while any two non-adjacent vertices are together adjacent to μ vertices.

A strongly regular graph of type (v, k, λ, μ) is usually denoted by $\mathrm{SRG}(v, k, \lambda, \mu)$.



A partition $\Pi = \{C_0, C_1, ..., C_{t-1}\}$ of the n vertices of a graph G is **equitable** (or regular) if for every pair of (not necessarily distinct) indices $i, j \in \{0, 1, ..., t-1\}$ there is a nonnegative integer $b_{i,j}$ such that each vertex $v \in C_i$ has exactly $b_{i,j}$ neighbors in C_j, regardless of the choice of v.

The $t \times t$ **quotient matrix** $B = [b_{i,j}]$ is well-defined if and only if the partition Π is equitable.

- C. E. Praeger, L. H. Soicher, Low rank representations and graphs for sporadic groups, Australian Mathematical Society Lecture Series 8, Cambridge University Press, Cambridge, 1997.

Let Γ be a SRG(v,k,λ,μ) and A be its adjacency matrix. Suppose an automorphism group G of Γ partitions the set of vertices V into t orbits $O_1,\ldots,O_t$, with sizes $n_1,\ldots,n_t$, respectively. This partition of V is equitable, and the quotient matrix $R=[r_{ij}]$ satisfy the following conditions

$$\sum_{j=1}^{t} r_{ij} = \sum_{i=1}^{t} \frac{n_i}{n_j} r_{ij} = k, \tag{3}$$

$$\sum_{s=1}^{t} \frac{n_s}{n_j} r_{si} r_{sj} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda-\mu) r_{ji}. \tag{4}$$

A $(t\times t)$-matrix $R=[r_{ij}]$ with entries satisfying conditions (3) and (4) is called an **orbit matrix for a strongly regular graph with parameters** (v,k,λ,μ) and orbit lengths distribution $(n_1,\ldots,n_t)$.

Orbit matrices of SRGs are used for a construction of SRGs in the following papers:

- M. Behbahani, C. Lam, Strongly regular graphs with non-trivial automorphisms, Discrete Math. 311 (2011), 132–144.
- D. Crnković, M. Maksimović, Construction of strongly regular graphs having an automorphism group of composite order, Contrib. Discrete Math. 15 (2020), 22–41.

A d-class **association scheme** on a finite non-empty set Ω is an ordered pair $(\Omega, \mathcal{R})$ with $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ a set of non-empty relations on Ω, such that the following axioms hold.

1. $\mathcal{R}$ is a partition of Ω^2.
2. R_0 is the identity relation.
3. For every relation $R_i \in \mathcal{R}$, its converse $R_i^T = \{(y, x) : (x, y) \in R_i\}$ is equal to R_i.
4. There are constants p_{ij}^k known as the **intersection numbers** of the association scheme $\mathcal{R}$, such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k.

Association schemes are introduced by Bose and Shimamoto in 1952.

Let $A_i = [a^i_{xy}]$ be the **adjacency matrix** of R_i, defined by $a^i_{xy} = 1$ if $(x, y) \in R_i$ and $a^i_{xy} = 0$ otherwise.

Each of the matrices A_i, $i \in \{1, 2, ..., d\}$, represents a simple graph Γ_i on the set of vertices Ω (if $(x, y) \in R_i$ then vertices x and y are adjacent in Γ_i). The graphs Γ_i form an edge-coloring of the complete graph on Ω.

An **equitable (or regular) partition** of an association scheme $(X, \mathcal{R})$ with d classes is a partition of X which is equitable with respect to each of the corresponding graphs Γ_i, $i \in \{1, 2, ..., d\}$.

Transitive action

A group G acting on a set Ω is said to be **transitive** on Ω if it has only one orbit, and so $G.\alpha = \Omega$ for all $\alpha \in \Omega$.
Equivalently, G is transitive if for every pair of points $\alpha, \beta \in \Omega$ there exist $x \in G$ such that $x.\alpha = \beta$.

Example

Symmetric group S_n acts transitively on $\Omega = \{1, \dots, n\}$, i.e. for all $x \in \Omega$, there exists an element $\sigma \in S_n$ such that $\sigma(1) = x$. Therefore, $S_n.1 = \Omega$.

- D. Crnković, V. Mikulić Crnković, A. Švob, On some transitive combinatorial structures constructed from the unitary group $U(3,3)$, J. Statist. Plann. Inference 144 (2014), 19–40.

Theorem

Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} \delta_i G_\alpha$, where $\delta_1, ..., \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, ..., \delta_s) = (\Omega_2, \mathcal{B})$ is a $1 - (n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}|} \sum_{i=1}^{s} |\alpha G_{\delta_i}|)$ design with $\frac{m \cdot |G_\alpha|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G/\bigcap_{x \in \Omega_2} G_x$ acts as an automorphism group on $(\Omega_2, \mathcal{B})$, transitively on points and blocks of the design. If $\Delta_2 = \Omega_2$ then the set $\mathcal{B}$ consists of one block, and $\mathcal{D}(G, \alpha, \delta_1, ..., \delta_s)$ is a design with parameters $1 - (n, n, 1)$.

If a group G acts transitively on Ω, $\alpha \in \Omega$, and Δ is an orbit of G_α, then $\Delta' = \{\alpha g \mid g \in G,\ \alpha g^{-1} \in \Delta\}$ is also an orbit of G_α. Δ' is called the orbit of G_α paired with Δ. It is obvious that $\Delta'' = \Delta$ and $|\Delta'| = |\Delta|$. If $\Delta' = \Delta$, then Δ is said to be self-paired.

Corollary

If $\Omega_1 = \Omega_2$ and Δ_2 is a union of self-paired and mutually paired orbits of G_α, then the design $\mathcal{D}(G, \alpha, \delta_1, ..., \delta_s)$ is a symmetric self-dual design and the incidence matrix of that design is the adjacency matrix of a $|\Delta_2|-$regular graph.

By applying Corollary we obtained distance-regular graphs from some simple groups, including the construction of first examples of strongly regular graphs with parameters $(216, 40, 4, 8)$ and $(540, 187, 58, 68)$ from the group $U(4, 2) \cong PSp(4, 3) \cong O^-(6, 2)$.

Example - Deza graph from $U(4,2)$ [D. Crnković, AŠ]

Combining incidence matrices of transitive 1-designs one can construct an adjacency matrix of a non-transitive graph.

The Zara graph with parameters (126,45,12,18) is a $SRG(126, 45, 12, 18)$ with the full automorphism group $U(4,3) \times Z_4$.

The group $U(4,2)$ acts on that Zara graph in three orbits of sizes 1, 45 or 80.

	1	45	80
1	0	45	0
45	1	12	32
80	0	18	27

The same action of $U(4,2)$ yields a strictly Deza graph (diameter 2, not a SRG) with parameters $(126, 45, 12, 18)$ and the full automorphism group $Z_2 \times (U(4,2) : Z_2)$.

Deza graphs were introduced in:

- M. Erickson, S. Fernando, W. H. Haemers, D. Hardy, J. Hemmeter, Deza graphs: A generalization of strongly regular graphs, J. Comb. Designs. 7 (1999), 395–405.

Deza graph

A Deza graph with parameters (v, k, b, a) is a k-regular graph with v vertices in which any two vertices have a or b $(a \leq b)$ common neighbours.

A Deza graph is strictly Deza if it has diameter 2, and is not strongly regular.

Up to isomorphism, there have been two known SRGs with parameters $SRG(81, 30, 12, 9)$: a graph Γ_1, being isomorphic to **the van Lint-Schrijver graph** and having full automorphism group of order 116640, and a second graph Γ_2 having full automorphism group of order 5832.

- D. Crnković, A. Švob, V. D. Tonchev, Strongly regular graphs with parameters $(81, 30, 9, 12)$ and a new partial geometry, J. Algebraic Combin. 53 (2021), 253–261.

The graphs Γ_1 and Γ_2 have the full automorphism groups G_1 or G_2 of order 116640 and 5832, respectively. New SRGs are constructed by presuming an action of a subgroup of G_1 or G_2.

- We used a method for finding strongly regular graphs based on **orbit matrices**, to show that there are exactly **three** nonisomorphic graphs SRG$(81, 30, 9, 12)$ which are invariant under a subgroup of order 360 of the automorphism group of Γ_1, and exactly **eleven** nonisomorphic graphs SRG$(81, 30, 9, 12)$ which are invariant under a subgroup of order 972 of the automorphism group of Γ_2.

Graph Γ	$\lvert Aut(\Gamma)\rvert$	Max. clique size of Γ
Γ_1	116640	6
Γ_2	5832	4
Γ_3	3888	6
Γ_4	1944	6
Γ_5	972	6
Γ_6	972	6
Γ_7	972	6
Γ_8	972	6
Γ_9	972	6
Γ_{10}	972	6
Γ_{11}	972	6
Γ_{12}	972	6
Γ_{13}	720	6
Γ_{14}	360	6

Table: Strongly regular graphs with parameters $(81, 30, 9, 12)$

SRGs admitting an action of a subgroup of $\mathrm{Aut}(\Gamma_2)$

There are exactly five conjugacy classes of subgroups of order 972 in the group $G_2 = \mathrm{Aut}(\Gamma_2)$ of order 5832, the representatives of which will be called $H_1^2, \ldots, H_5^2$.

The groups H_1^2 and H_2^2 **act transitively on 81 vertices** and produce **two** non-isomorphic strongly regular graphs, Γ_1 and Γ_2 of orders 116640 and 5832, respectively.

The group H_3^2 is acting on the set of vertices of Γ_2 in three orbits of length 27 and with the orbit matrix OM_3^2.

From the orbit matrix OM_3^2 we obtained **two** non-isomorphic strongly regular graphs, the graph Γ_2 and a new graph Γ_4 with the full automorphism group of order 1944.

$$OM_3^2 = \begin{pmatrix} 12 & 9 & 9 \\ 9 & 12 & 9 \\ 9 & 9 & 12 \end{pmatrix}$$

The groups H_4^2 and H_5^2 are acting in the same way, with two orbits, one of size 27 and other of size 54 and with the orbit matrix OM_4^2 given below. From the group H_4^2 we obtained **four** non-isomorphic strongly regular graphs, one of them is Γ_2. Two of these SRGs, Γ_5 and Γ_6, have the full automorphism groups of order 972, and Γ_4 with the full automorphism group of order 1944.

From the group H_5^2 we obtained 12 non-isomorphic strongly regular graphs, among them Γ_1, Γ_2 and Γ_4. Eight of these SRGs, denoted by $\Gamma_5, \dots, \Gamma_{12}$, have the full automorphism groups of order 972, and Γ_3 has the full automorphism group of order 3888.

$$OM_4^2 = \begin{pmatrix} 12 & 18 \\ 9 & 21 \end{pmatrix}$$

Partial geometry

A **partial geometry** with parameters s, t, α, or shorty, $pg(s, t, \alpha)$, is a pair (P, L) of a set P of **points** and a set L of *lines*, with an incidence relation between points and lines, satisfying the following axioms:

1. A pair of distinct points is not incident with more than one line.
2. Every line is incident with exactly $s + 1$ points ($s \geq 1$).
3. Every point is incident with exactly $t + 1$ lines ($t \geq 1$).
4. For every point p not incident with a line l, there are exactly α lines ($\alpha \geq 1$) which are incident with p, and also incident with some point incident with l.

- R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs. *Pacific J. Math.* **13** (1963), 389–419.

In terms of s, t, α, the number $v = |P|$ of points, and the number $b = |L|$ of lines of a partial geometry $pg(s, t, \alpha)$ are given by:

$$v = \frac{(s+1)(st+\alpha)}{\alpha}, \;\; b = \frac{(t+1)(st+\alpha)}{\alpha}. \qquad (5)$$

If $G = (P, L)$ is a partial geometry $pg(s, t, \alpha)$, the incidence structure G' having as points the lines of G, and having as lines the points of G, where a point and a line are incident in G' if and only if the corresponding line and a point of G are incident, is a partial geometry $pg(t, s, \alpha)$, called the **dual** of G.

If $G = (P, L)$ is a partial geometry $pg(s, t, \alpha)$ with point set P and line set L, the **point graph** Γ_P of G is the graph with vertex set P, where two vertices are adjacent if the corresponding points of G are collinear.

The **line graph** Γ_L of G is the graph having as vertices the lines of G, where two lines are adjacent if they share a point.

Both Γ_P and Γ_L are **strongly regular graphs** (Bose).

The parameters v, k, λ, μ of Γ_P are given by:

$$v = (s+1)(st+\alpha)/\alpha, k = s(t+1), \lambda = s-1+t(\alpha-1), \mu = \alpha(t+1). \quad (6)$$

A partial geometry $pg(s, t, \alpha)$ with $\alpha = s + 1$ is a Steiner 2-$(v, s + 1, 1)$ design, or dually, if $\alpha = t + 1$, then the dual geometry is a Steiner 2-$(b, t + 1, 1)$ design. If $\alpha = s$, or dually, $\alpha = t$, then G is a net of order $s + 1$ and degree $t + 1$. A partial geometry with $\alpha = 1$ is a generalized quadrangle.

A partial geometry $pg(s, t, \alpha)$ is called **proper** if $1 < \alpha < min(s, t)$.

A strongly regular graph Γ whose parameters n, k, λ, μ can be written as in eq. 6 for some integers s, t, α is called **pseudo-geometric**, and Γ is called **geometric** if there exists a partial geometry G with parameters s, t, α such that Γ is the point graph of G.

Bose, 1963.

A pseudo-geometric strongly regular graph Γ with parameters (6) is geometric if and only if there exists a set S of $b = (t+1)(st+\alpha)/\alpha$ cliques of size $s+1$ such that every two cliques from S share at most one vertex.

The known proper partial geometries are divided into eight types, four of which are infinite families, and there are four sporadic geometries that do not belong to any known infinite family.

- J. A. Thas, Partial Geometries, in: C. J. Colbourn, J. H. Dinitz (Eds.), Handbook of Combinatorial Designs, 2^{nd} ed., Chapman & Hall/CRC, Boca Raton, 2007, pp. 557–561.

One of the four sporadic examples is a partial geometry $pg(5,5,2)$ discovered by van Lint and Schrijver in 1981:

- J. H. van Lint, A. Schrijver, Construction of strongly regular graphs, two-weight codes and partial geometries by finite fields, Combinatorica 1 (1981), 63–73.

The point graph of the van Lint-Schrijver geometry has parameters $v = 81,\ k = 30, \lambda = 9, \mu = 12$, and is invariant under the elementary abelian group of order 81 acting regularly on the set of vertices.

The only previously known partial geometry $pg(5,5,2)$ is the one constructed by van Lint. This partial geometry has the full automorphism group of order 58320 and the corresponding SRG is Γ_1 with the full automorphism group G_1 of order 116640.

Twelve new strongly regular graphs with parameters $(81,30,9,12)$ are found as graphs invariant under certain subgroups of the automorphism groups of the two previously known graphs that arise from 2-weight codes.

One of these new graphs is geometric and yields a partial geometry with parameters $pg(5,5,2)$ that is not isomorphic to the partial geometry discovered by J. H. van Lint and A. Schrijver in 1981.

Graph Γ	$\lvert Aut(\Gamma)\rvert$	Max. clique size of Γ
Γ_1	116640	6
Γ_2	5832	4
Γ_3	3888	6
Γ_4	1944	6
Γ_5	972	6
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Γ_8	972	6
Γ_9	972	6
Γ_{10}	972	6
Γ_{11}	972	6
Γ_{12}	972	6
Γ_{13}	720	6
Γ_{14}	360	6

Table: Strongly regular graphs with parameters $(81, 30, 9, 12)$

The graph Γ_{12} has 108 6-cliques, 81 of them form the lines of the new $pg(5,5,2)$.

This new partial geometry $pg(5,5,2)$ has the full automorphism group of order 972, which acts on the points and the blocks of the $pg(5,5,2)$ in two orbits of lengths 27 and 54.

The new $pg(5,5,2)$ is self-dual, i.e. its line graph is isomorphic to its point graph.

Note: An isomorphic partial geometry was simultaneously and independently constructed by V. Krčadinac by using another method based on geometrical structures.

Thank you for your attention!