Eigenvalue bounds for the independence and chromatic number of graph powers

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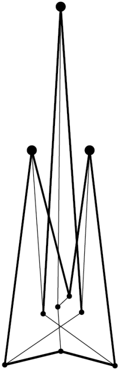
Outline

1. Background
2. Eigenvalue bounds: an overview
3. New inertia and ratio-type bounds and optimization

Background

Spectral graph theory

A graph, its adjacency matrix and its spectrum:



$$\Longleftrightarrow \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$\Downarrow$ $\Uparrow$???

spectrum (eigenvalues): $3, 1, 1, 1, 1, 1, -2, -2, -2, -2$

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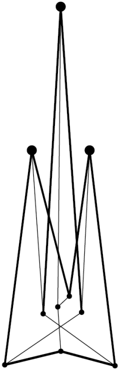

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Adjacency matrix $A = (a_{ij})$

Power adjacency matrix $A^k = (a_{ij}^k)$

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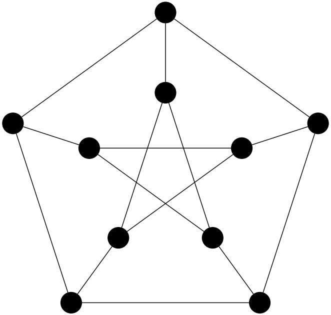
algebraic combinatorics

Eigenvalues

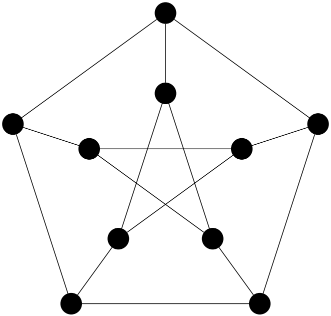
Spectrum: $\lambda_1 \geq \cdots \geq \lambda_n$



$$3, 1, 1, 1, 1, 1, -2, -2, -2, -2$$

Eigenvalues

Spectrum: $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$



$3^1, 1^5, -2^4$

Walk-regularity and k-partially walk-regularity

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  (Fiol and Garriga 1998) If G is k-partially walk-regular, for any polynomial $p \in \mathbb{R}_k[x]$, the diagonal of $p(A)$ is constant with entries

$$(p(A))_{uu} = \frac{1}{n} \operatorname{tr} p(A) = \frac{1}{n} \sum_{i=1}^{n} p(\lambda_i) \quad \text{for all } u \in V.$$

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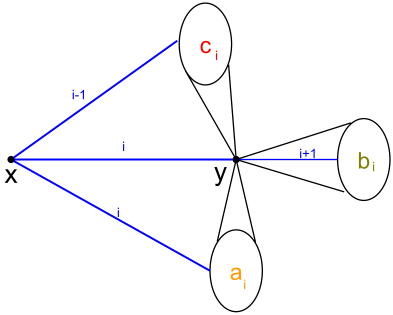
  Every graph is k-partially walk-regular for $k = 0, 1$, and every regular graph is 2-partially walk-regular.

- G is k-partially walk-regular for any k iff G is walk-regular.

Distance-regularity and k-partially distance-regularity

- A graph G of diameter D is **distance-regular** if there are constants c_i, a_i, b_i such that for all $i = 0, 1, \ldots, D$, and all vertices x and y at distance $i = d(x, y)$, among the neighbors of y, there are c_i at distance $i - 1$ from x, a_i at distance i, and b_i at distance $i + 1$.



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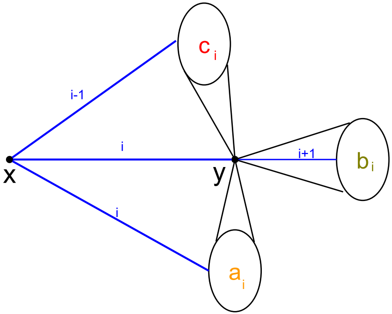

- G is k**-partially distance-regular** if it is distance-regular up to distance k.

Graph powers

The k^{th} **power of a graph** $G = (V, E)$, denoted by G^k, is formed by connecting two vertices if they are at distance at most k.

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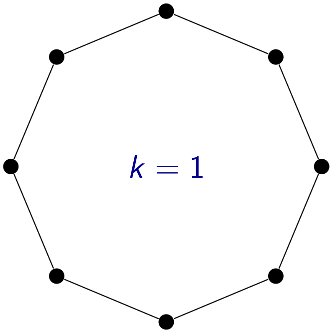

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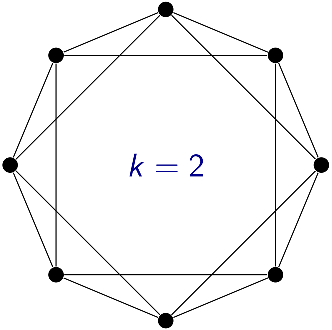

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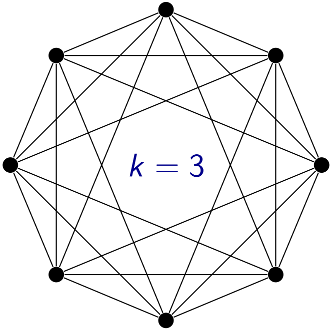

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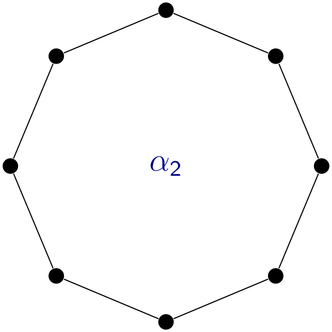

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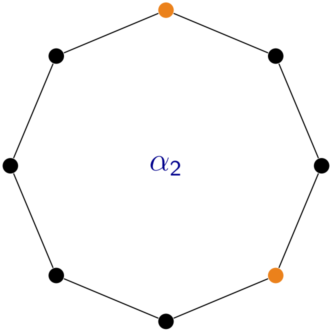

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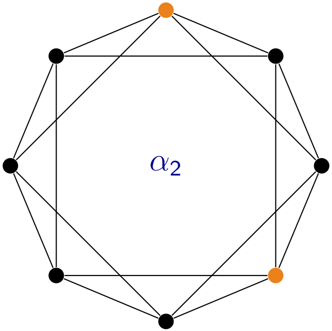

Note: $\alpha_k(G) = \alpha(G^k)$

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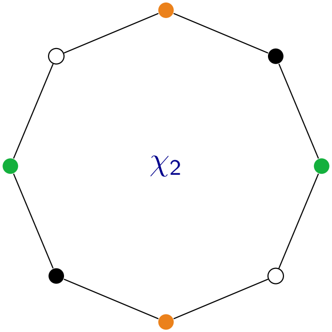

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Upper bounds on α_k give lower bounds on χ_k and vice versa:

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$$

Applications α_k

- Coding theory: codes relate to k-independent sets in Hamming graphs, and eigenvalue bounds on α_k have been used to show the non-existence of perfect codes (Fiol 2020).

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  We use MILPs to compute when the new inertial-type bound is tight.
- Related to other graph parameters: α_k has been used to obtain tight lower bounds for the average distance (Firby and Haviland 1997), ...

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Examples:

- average degree (Devos, McDonald and Scheide 2013)
- rainbow number (Basavaraju, Chandran, Rajendraprasad and Ramaswamy 2014)
- eigenvalues
- ...

Motivation α_k

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No, in general the spectrum of G^k cannot be derived from G, and vice versa.

$$\Downarrow$$

We will find bounds that only depend on the spectrum of G.

Overall goal

Extend two classic eigenvalue bounds for α to α_k in terms of the eigenvalues of the original graph.

Main tool: interlacing

Let $m < n$.
Sequences $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_m$ **interlace** if

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \qquad (1 \leq i \leq m)$$

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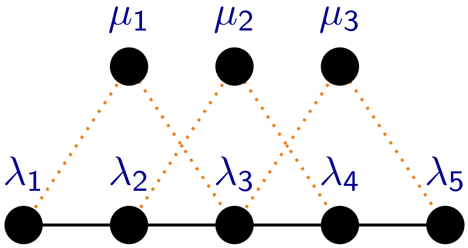

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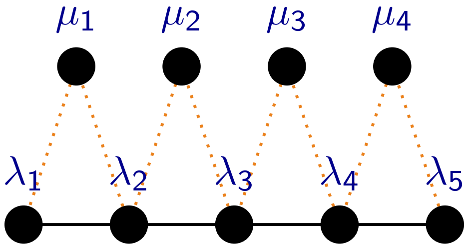

$m = n - 1$

Eigenvalue interlacing

$\lambda_1, \lambda_2, \ldots, \lambda_n$ eigenvalues of a matrix A

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$\mu_1, \mu_2, \ldots, \mu_m$ eigenvalues of a matrix B

First case of eigenvalue interlacing

1. B is a principal submatrix of A.

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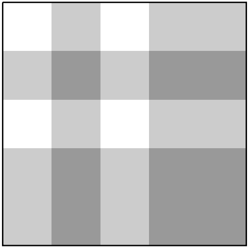
(Cauchy interlacing)
If B is a principal submatrix of A, then the eigenvalues of B interlace those of A.



Second case of eigenvalue interlacing

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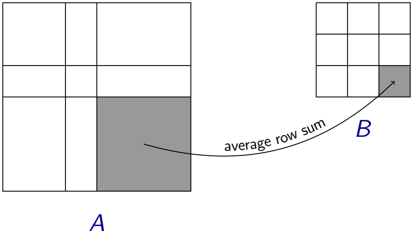
(Haemers interlacing 1995)
If B is the quotient matrix of a partition of A, then the eigenvalues of B interlace the eigenvalues of A.



Eigenvalue bounds: an overview

Classic eigenvalue bound I: inertia

Inertia bound (Cvetković 1972)

If G is a graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$\alpha(G) \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

Classic eigenvalue bound II: ratio

Ratio bound (Hoffman 1970)

If G is regular with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$\alpha(G) \leq n\frac{-\lambda_n}{\lambda_1 - \lambda_n}$$

and if an independent set C meets this bound then every vertex not in C is adjacent to precisely $-\lambda_n$ vertices of C.

⚠ Delsarte proved the ratio bound for SRGs, later Hoffman extended it to regular graphs and Haemers to irregular graphs.

(Lovász 1979)

The Lovász theta number $\vartheta(G)$ is a lower bound for the Hoffman bound.

More on the ratio bound

ELSEVIER

Linear Algebra and its Applications
Volume 617, 15 May 2021, Pages 215-219

Hoffman's ratio bound

In memory of Alan J. Hoffman

Willem H. Haemers

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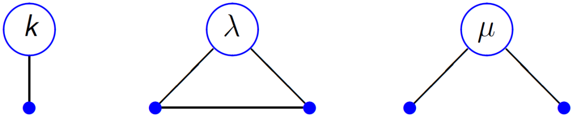
https://doi.org/10.1016/j.laa.2021.02.010

Abstract

Hoffman's ratio bound is an upper bound for the independence number of a regular graph in terms of the eigenvalues of the adjacency matrix. The bound has proved to be very useful and has been applied many times. Hoffman did not publish his result, and for a great number of users the emergence of Hoffman's bound is a black hole. With his note I hope to clarify the history of this bound and some of its generalizations.

Inertia vs ratio bound for some strongly regular graphs



Graph	(n, k, λ, μ)	α	Inertia bound	(Floor of) ratio bound
Cycle C_5	$(5, 2, 0, 1)$	2	2	2
Petersen	$(19, 3, 0, 1)$	4	4	4
Clebsh	$(16, 5, 0, 2)$	5	**5**	**6**
Hoffman-Singleton	$(50, 7, 0, 1)$	15	21	15
Gewirtz	$(56, 10, 0, 2)$	16	20	16
Mesner M_{22}	$(77, 16, 0, 7)$	21	21	21
Higman-Sims	$(100, 22, 0, 6)$	22	**22**	**26**

Some known upper bounds on α_k

- (Firby and Haviland 1997) For connected graphs using average distance.
- (Fiol 1997) For regular graphs using eigenvalues and alternating polynomials.
- (Atkinson and Frieze 2003) For random graphs $G_{n,p}$, $p = d/n$ (d a large constant).
- (Beis, Duckworth and Zito 2005) For random r-regular graphs.
- (O, Shi and Taoqiu 2019) For r-regular graphs for every $k \geq 2$ and $r \geq 3$.
- (Jou, Lin and Lin 2020) For trees and $k = 2$.

Optimization and eigenvalue bounds

Independence number:

- (Delsarte 1973) LP bound on α for distance-regular graphs.
- (Lovász 1979) SDP bound ϑ.
- ...

***k*-independence number:**

Ratio bound	Inertia bound
$\downarrow$	$\downarrow$
(Fiol 2019) LP with minor polynomials	?

Line of work

(1) (A., Cioabă and Tait 2016) New bounds on α_k in terms of λ_i^k.

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Which polynomial gives the best bound for a specific graph?

(3) (A., Coutinho, Fiol, Nogueira and Zeijlemaker 2022) Optimize the bounds over $p \in \mathbb{R}_k[x]$.

Optimization of the new eigenvalue bounds for the independence and chromatic number of graph powers

Joint work with G. Coutinho, M.A. Fiol, B. Nogueira and S. Zeijlemaker

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Graphs with large chromatic number

Question (Alon and Mohar 2000)

What is the largest possible value of the chromatic number $\chi(G^k)$ of G^k, among all graphs G with maximum degree at most d and girth at least g?

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- $k = 1$: long-standing problem by Vizing, settled asymptotically by (Johansson 1996) using the probabilistic method.
- $k = 2$: settled asymptotically by (Alon and Mohar 2002).
- $k \geq 3$: bounds by (Alon and Mohar 2002), (Kang and Pirot 2016), (Kang and Pirot 2018), ...

A lower bound on χ_k

Let $G = (V, E)$ be a graph with spectrum $\theta_0^{m_0}, \ldots, \theta_d^{m_d}$ and consider the inner product

$$\langle f, g \rangle_G = \frac{1}{n} \operatorname{tr}(f(A)g(A)) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\theta_i) g(\theta_i).$$

The **predistance polynomials** $p_0, \ldots, p_d$ are orthogonal polynomials with respect to the above product, with $\operatorname{dgr} p_i = i$, and normalized such that $\|p_i\|_G^2 = p_i(\theta_0)$ (Fiol and Garriga 1997).

A lower bound on χ_k

Let $G = (V, E)$ be a graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and predistance polynomials $p_0, \ldots, p_d$.
For a given integer $k \leq d$, consider the polynomial $q_k = p_0 + \cdots + p_k$.

(Fiol 2012)

Let $s_k(u)$ be the number of vertices at distance at most k from u. Then $q_k(\lambda_1)$ is bounded above by

$$q_k(\lambda_1) \leq H_k = \frac{n}{\sum_{i \in V} \frac{1}{s_k(u)}}.$$

Equality occurs if and only if $q_k(A) = I + A(G^k)$.

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$\Rightarrow$ Spectrum of G and G^k are related.

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(A., Coutinho, Fiol, Nogueira and Zeijlemaker 2022)

Let $q'_k = q_k - 1$. If G is regular with eigenvalues satisfying $q_k(\lambda_1) = H_k$, then

$$\chi_k \geq \frac{n}{\min\{|\{i : q'_k(\lambda_i) \geq 0\}|, |\{i : q'_k(\lambda_i) \leq 0\}|\}}$$

and

$$\chi_k \geq \frac{n}{1 - \frac{q'_k(\lambda_1)}{\min\{q'_k(\lambda_i)\}}}.$$

The spectra of G and G^k are related

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First spectral bounds for Alon and Mohar question for regular graphs. But how do we find the polynomial $q_k = p_0 + \cdots + p_k$?

(A., van Dam and Fiol 2016)

$q_k(A) = A(G^k) + I$ when G is a δ-regular graph with girth g and $k = \lfloor \frac{g-1}{2} \rfloor$. In this case G is k-partially distance-regular, and

$$q_0 = 1,\ q_1 = 1 + x,\ q_{i+1} = xq_i - (\delta - 1)q_{i-1}.$$

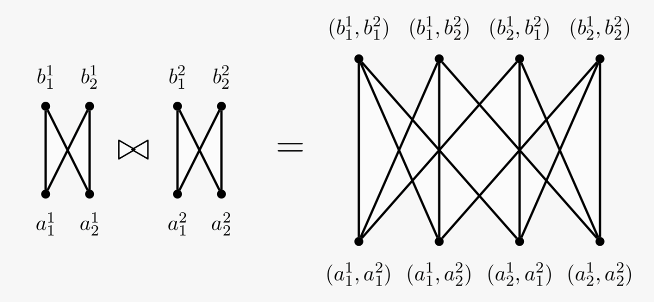
Tight examples

Our bound is tight for several named Sage graphs.

Name	Girth g	$k = \lfloor \frac{g-1}{2} \rfloor$	α_k
Moebius-Kantor graph	6	2	4
Nauru graph	6	2	6
Blanusa First Snark graph	5	2	4
Blanusa Second Snark graph	5	2	4
Brinkmann graph	5	2	3
Heawood graph	6	2	2
Sylvester graph	5	2	6
Coxeter graph	7	3	4
Dyck graph	6	2	8
F26A graph	6	2	6
Flower Snark graph	5	2	5

Tight examples

(Kang and Pirot 2016) used **balanced bipartite products** $\bowtie$ for their lower bound construction.



This product also gives several graphs which attain equality for our bound, for example the products of even cycles
$C_8 \bowtie C_8$, $C_8 \bowtie C_{12}$, ...

The spectrum of G^k and G are not related.

The spectrum of G^k and G are not related.
Optimization of inertial type bounds

Why optimization using MILPs?

(i) The quantum k-independence number is upper bounded by the inertial-type bound (A., Elphick and Wocjan 2022):

$$\alpha_k \leq \alpha_{kq} \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}.$$

For $k > 1$ we can use the MILPs to compute values of the quantum parameter when the bound is tight. For $k = 1$:

$$\alpha \leq \alpha_q \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

(ii) Closed formulas for small k.

(iii) Use the polynomials involved in the MILPs:
inertial-type bound THIS TALK
ratio-type bound (Fiol 2020)
to relate both bounds (A. Dalfó, Fiol, Zeijlemaker 2023+)

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First inertial-type bound

Let G be a graph with adjacency matrix A and $p \in \mathbb{R}_k[x]$.

$$w(p) := \min_i \ p(A)_{ii}$$

$$W(p) := \max_i \ p(A)_{ii}$$

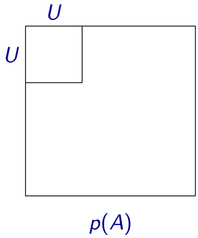
(A., Coutinho, Fiol 2019)

Let $p \in \mathbb{R}_k[x]$, then

$$\alpha_k(G) \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}.$$

Proof sketch

Let U be a k-independent set of G with size α_k.



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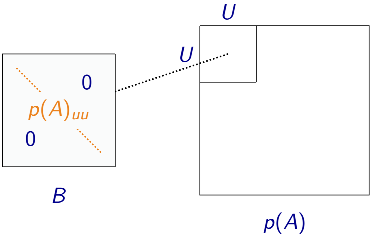

Let μ be the smallest eigenvalue of B.

- Cauchy interlacing ($\lambda_i \geq \mu_i$ for $i = 1, \ldots, m = |U|$):
  $\geq |U|$ eigenvalues of $p(A)$ are larger than μ
- $\mu \geq w(p)$ by definition of $w(p) = \min_{u \in V}\{(p(A))_{uu}$.

Therefore, $|U| \leq |\{i : p(\lambda_i) \geq w(p)\}|$.

First inertial-type bound: corollary

For $k = 1$,

First inertial-type bound: corollary

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Inertia bound (Cvetković 1972)

If G is a graph, then

$$\alpha(G) \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

First inertial-type bound: optimization

$$\alpha_k(G) \leq \min\{|i : p_k(\lambda_i) \geq w(p_k)|, |i : p_k(\lambda_i) \leq W(p_k)|\}$$

Linear?

First inertial-type bound: optimization

$$\alpha_k(G) \leq \min\{|i : p_k(\lambda_i) \geq w(p_k)|, |i : p_k(\lambda_i) \leq W(p_k)|\}$$

Linear?

Invariant under scaling and translation

- may assume $\min\{|i : p_k(\lambda_i) \geq w(p_k)|\}$, otherwise take $-p_k$
- translate: $\min\{|i : p_k(\lambda_i) \geq 0|\}$.

First MILP

$\alpha_k \leq \min\{|i : p_k(\lambda_i) \geq 0|\}$

Let G be a graph with spectrum $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$ and $p_k(x) = a_k x^k + \cdots + a_0$ the polynomial to optimize.

For each $u \in V$, assume $w(p_k) = p_k(A)_{uu}$ and solve

$$
\begin{array}{rl}
\texttt{minimize} & \boldsymbol{m}^T \boldsymbol{b} \\
\texttt{subject to} & \sum_{i=0}^{k} a_i (A^i)_{vv} \geq 0, \quad v \in V(G) \backslash \{u\} \\
 & \sum_{i=0}^{k} a_i (A^i)_{uu} = 0 \\
 & \sum_{i=0}^{k} a_i {\theta_j}^i - M b_j + \varepsilon \leq 0, \quad j = 0, ..., d \\
 & \boldsymbol{b} \in \{0,1\}^{d+1}
\end{array}
$$

with M large, $\varepsilon > 0$ small.

variables: $a_1, \ldots, a_k$, $(b_0, \ldots, b_d)$
parameters: k, $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$

First MILP $\alpha_k \leq \min\{|i : p_k(\lambda_i) \geq 0|\}$

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Vector b encodes whether $p_k(\theta_i) \geq w(p_k)$: $b_i = 1$ iff $p_k(\theta_i) \geq 0$.

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Let G be a graph with spectrum $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$ and $p_k(x) = a_k x^k + \cdots + a_0$ the polynomial to optimize.

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with M large, $\varepsilon > 0$ small.

⚠ Linear combination of the eigenvalues mutiplicities (minimizing the quantity of indices j)

First MILP: results

For large graphs, solving n MILPs takes a lot of time. However, this first inertial-type bound does not require walk-regularity like in the optimization of the ratio-type bound by (Fiol 2020).

Proportion of small irregular graphs for which the optimal solution of the MILP equals α_2:

Number of vertices	4	5	6	7	8	9
Proportion	0.86	0.84	0.76	0.62	0.46	0.27

First MILP: results

Name	Best 2019	ϑ_2	First MILP	α_2
Balaban 10-cage	17	17	19	17
Frucht graph	3	3	3	3
Meredith Graph	14	10	10	10
Moebius-Kantor Graph	4	4	6	4
Bidiakis cube	3	2	4	2
Gosset Graph	2	2	8	2
Gray graph	14	11	19	11
Nauru Graph	6	5	8	6
Blanusa First Snark Graph	4	4	4	4
Pappus Graph	4	3	7	3
Blanusa Second Snark Graph	4	4	4	4
Poussin Graph	-	2	4	2
Brinkmann graph	4	3	6	3
Harborth Graph	12	9	13	10
Perkel Graph	10	5	18	5
Harries Graph	17	17	18	17
Bucky Ball	16	12	16	12
Harries-Wong graph	17	17	18	17
Robertson Graph	3	3	5	3
Heawood graph	3	2	2	2
Herschel graph	-	2	3	2
Hoffman Graph	3	2	5	2
...				

First inertial-type bound: walk-regular graphs

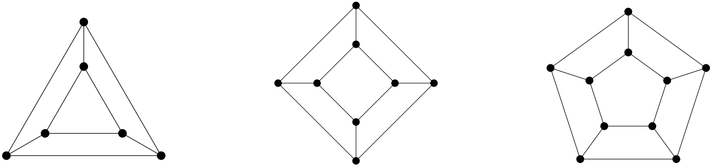
Let G be a k-partially walk-regular. Then $p_k(A)$ has constant diagonal (all vertices have the same number of closed walks of length smaller of equal than k), so we only have to run the MILP once:

$$
\begin{array}{rl}
\texttt{minimize} & \boldsymbol{m}^T\boldsymbol{b} \\
\texttt{subject to} & \sout{\sum_{i=0}^{k} a_i(A^i)_{vv} \geq 0, \quad v \in V(G)\backslash\{u\}} \\
& \sout{\sum_{i=0}^{k} a_i(A^i)_{uu} = 0} \\
& \sum_{i=0}^{d} m_i p_k(\theta_i) = 0 \\
& \sum_{i=0}^{k} a_i {\theta_j}^i - Mb_j + \varepsilon \leq 0, \quad j = 0, \ldots, d \\
& \boldsymbol{b} \in \{0,1\}^{d+1}
\end{array}
$$

first constraint can be removed if we use as p the predistance polynomials

First inertial-type bound: equality $k = 2$

Prism graphs Γ_n



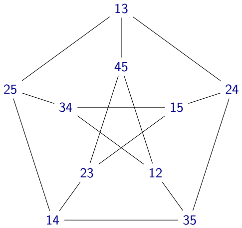
Γ_3 Γ_4 Γ_5

$$\alpha_2(\Gamma_{4i+j}) = \begin{cases} 2i+1 & \text{if } j = 3 \\ 2i & \text{otherwise} \end{cases}$$

These graphs are walk-regular. For $n \neq 2 \mod 4$, the MILP is tight.

First inertial-type bound: equality

Odd graphs O_ℓ: vertices corresponding to the $(\ell-1)$-subsets of a $(2\ell-1)$-set, and the adjacencies are defined by void intersection.



O_3

(A., Coutinho, Fiol, Nogueira and Zeijlemaker 2022)

For $i = 0, \ldots, \ell-1$, let μ_i and m_i be the eigenvalues and multiplicities of the Odd graph $O_\ell = O_{d+1}$. Then,

$$\alpha_{d-1}(O_{d+1}) \leq \left\{ \begin{array}{ll} m_1 & \text{for even } d \\ m_1+1 & \text{for odd } d \end{array} \right\} = \left\{ \begin{array}{ll} 2d & \text{for even } d, \\ 2d+1 & \text{for odd } d. \end{array} \right.$$

First inertial-type bound: equality

Odd graph $O_\ell = O_{d+1}$	α_{d-1}	First inertial-type bound
O_2 (K_3)	$\alpha_0 = 3$	$m_0 + m_1 = 3$
O_3 (Petersen)	$\alpha_1 = 4$	$m_1 = 4$
O_4	$\alpha_2 = 7$	$m_0 + m_1 = 7$
O_5	$\alpha_3 = 7$	$m_1 = 8$
O_6	$\alpha_4 = 11$	$m_0 + m_1 = 11$
O_7	$\alpha_5 = 12$	$m_1 = 12$
O_8	$\alpha_6 = 15$	$m_0 + m_1 = 15$
O_9	$\alpha_7 = 15$	$m_1 = 16$
O_{10}	$\alpha_8 = 19$	$m_0 + m_1 = 19$
O_{11}	$\alpha_9 = 19$	$m_1 = 20$
O_{12}	$\alpha_{10} = 23$	$m_0 + m_1 = 23$
O_{14}	$\alpha_{12} = 27$	$m_0 + m_1 = 27$
O_{16}	$\alpha_{14} = 31$	$m_0 + m_1 = 31$

Table: The known exact values of α_{d-1} and the upper bounds for the Odd graphs O_{d+1}.

Second inertial-type bound

Sometimes the first inertial bound can be strengthened, using a different and more sophisticated proof strategy:

(A., Coutinho, Fiol, Nogueira and Zeijlemaker 2022)

Let G be a k-partially walk-regular graph with adjacency matrix eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, $p_k \in \mathbb{R}_k[x]$ such that $\sum_{i=1}^n p_k(\lambda_i) = 0$. Then,

$$\chi_k \geq 1 + \max\left(\frac{|j : p_k(\lambda_j) < 0|}{|j : p_k(\lambda_j) > 0|}\right).$$

Second MILP

$$\chi_k \geq 1 + \max\left(\frac{|j : p_k(\lambda_j) < 0|}{|j : p_k(\lambda_j) > 0|}\right)$$

variables: $a_1, \ldots, a_k$, $(b_1, \ldots, b_n)$, $(c_1, \ldots, c_n)$
parameters: k, $\lambda_1, \ldots, \lambda_n$

$$\begin{array}{rll}
\texttt{maximize} & 1 + \frac{n - \mathbf{1}^T \boldsymbol{b}}{\ell} & \\
\texttt{subject to} & \sum_{j=1}^{n} \sum_{i=0}^{k} a_i \lambda_j^i = 0 & \\
& \sum_{i=0}^{k} a_i \lambda_j^i - M b_j + \varepsilon \leq 0, & j = 1, \ldots, n \\
& \sum_{i=0}^{k} a_i \lambda_j^i - M c_j \leq 0, & j = 1, \ldots, n \\
& \sum_{i=1}^{n} c_i = \ell & \\
& \boldsymbol{b} \in \{0,1\}^n, \quad \boldsymbol{c} \in \{0,1\}^n &
\end{array}$$

⚠ Now we look at all eigenvalues, including the repeated ones

Second MILP

$$\chi_k \geq 1 + \max\left(\frac{|j : p_k(\lambda_j) < 0|}{|j : p_k(\lambda_j) > 0|}\right)$$

- Trace condition $\operatorname{tr} p_k(A) = 0$
- If $p_k(\lambda_j) \geq 0$, then $b_j = 1$. If $p_k(\lambda_j) > 0$, then $c_j = 1$
- Fix $\sum c_i = \ell$, solve for $\ell = 1, \ldots, n-1$
- Maximize $|j : p_k(\lambda_j) < 0| = n - 1^T \boldsymbol{b}$

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\end{array}$$

Second MILP

$$\chi_k \geq 1 + \max\left(\frac{|j{:}p_k(\lambda_j)<0|}{|j{:}p_k(\lambda_j)>0|}\right)$$

- Trace condition $\operatorname{tr} p_k(A) = 0$
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- Maximize $|j : p_k(\lambda_j) < 0| = n - 1^T \boldsymbol{b}$

$$\begin{array}{rll}
\texttt{maximize} & 1 + \frac{n - 1^T \boldsymbol{b}}{\ell} & \\
\texttt{subject to} & \sum_{j=1}^{n} \sum_{i=0}^{k} a_i \lambda_j^i = 0 & \\
& \sum_{i=0}^{k} a_i \lambda_j^i - M b_j + \varepsilon \leq 0, & j = 1, \ldots, n \\
& \sum_{i=0}^{k} a_i \lambda_j^i - M c_j \leq 0, & j = 1, \ldots, n \\
& \sum_{i=1}^{n} c_i = \ell & \\
& \boldsymbol{b} \in \{0,1\}^n, \quad \boldsymbol{c} \in \{0,1\}^n &
\end{array}$$

Second MILP

$$\chi_k \geq 1 + \max\left(\frac{|j : p_k(\lambda_j) < 0|}{|j : p_k(\lambda_j) > 0|}\right)$$

- Trace condition $\operatorname{tr} p_k(A) = 0$
- If $p_k(\lambda_j) \geq 0$, then $b_j = 1$. If $p_k(\lambda_j) > 0$, then $c_j = 1$
- Fix $\sum c_i = \ell$, solve for $\ell = 1, \ldots, n-1$
- Maximize $|j : p_k(\lambda_j) < 0| = n - 1^T \boldsymbol{b}$

$$\begin{array}{ll}
\texttt{maximize} & 1 + \frac{n - 1^T \boldsymbol{b}}{\ell} \\
\texttt{subject to} & \sum_{j=1}^{n} \sum_{i=0}^{k} a_i \lambda_j^i = 0 \\
& \sum_{i=0}^{k} a_i \lambda_j^i - M b_j + \varepsilon \leq 0, \quad j = 1, \ldots, n \\
& \sum_{i=0}^{k} a_i \lambda_j^i - M c_j \leq 0, \quad j = 1, \ldots, n \\
& \sum_{i=1}^{n} c_i = \ell \\
& \boldsymbol{b} \in \{0,1\}^n, \quad \boldsymbol{c} \in \{0,1\}^n
\end{array}$$

Second MILP: results

Name	Best 2019	ϑ_2	First MILP	Second MILP	α_2
Balaban 10-cage	17	17	19	19	17
Frucht graph	3	3	3	3	3
Meredith Graph	14	10	10	10	10
Moebius-Kantor Graph	4	4	6	4	4
Bidiakis cube	3	2	4	3	2
Gosset Graph	2	2	8	2	2
Gray graph	14	11	19	19	11
Nauru Graph	6	5	8	8	6
Blanusa First Snark Graph	4	4	4	4	4
Pappus Graph	4	3	7	6	3
Blanusa Second Snark Graph	4	4	4	4	4
Brinkmann graph	4	3	6	6	3
Harborth Graph	12	9	13	13	10
...					
Klein 7-regular Graph	3	3	9	3	3
...					

Tight families:

- Prism graphs Γ_n with $n \neq 2 \mod 4$
- Incidence graphs of projective planes $PG(2;q)$ (q prime power)

Ratio-type bound

$W(p) := \max_{u\in V}\{(p(A))_{uu}\}$
$w(p) := \min_{u\in V}\{(p(A))_{uu}\}$
$\lambda(p) := \max_{i\in[2,n]}\{p(\lambda_i)\}$

(A., Coutinho, Fiol 2019)

Let G be a regular graph with n vertices and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $p \in \mathbb{R}_k[x]$ with corresponding parameters $W(p)$ and $\lambda(p)$, and assume $p(\lambda_1) > \lambda(p)$. Then,

$$\alpha_k \leq n\frac{W(p) - \lambda(p)}{p(\lambda_1) - \lambda(p)}.$$

Ratio-type bound: corollary

For $k = 1$,

Ratio-type bound: corollary

For $k = 1$,

Ratio bound (Hoffman 1970)

If G is regular then

$$\alpha(G) \leq n \frac{-\lambda_n}{\lambda_1 - \lambda_n}.$$

Ratio-type bound: optimization

(Fiol 2020)

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Ratio-type bound: optimization

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For k-partially walk-regular graphs

$$(p(A))_{uu} = \frac{1}{n} \operatorname{tr} p(A) = \frac{1}{n} \sum_{i=1}^{n} p(\lambda_i) \quad \text{for all } u \in V.$$

Why optimization using MILPs?

(i) The quantum k-independence number is upper bounded by the inertial-type bound (A., Elphick and Wocjan 2022):

$$\alpha_k \leq \alpha_{kq} \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}.$$

For $k > 1$ we can use the MILPs to compute values of the quantum parameter when the bound is tight. For $k = 1$:

$$\alpha \leq \alpha_q \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

(ii) Closed formulas for small k.

(iii) Use the polynomials involved in the MILPs:
inertial-type bound THIS TALK
ratio-type bound (Fiol 2020)
to relate both bounds (A. Dalfó, Fiol, Zeijlemaker 2023+)

Ratio-type bound: best polynomial for k=2

(A., Coutinho, Fiol 2019)

Let G be a δ-regular graph with n vertices and distinct eigenvalues $\theta_0(=\delta) > \theta_1 > \cdots > \theta_d$ with $d \geq 2$. Let θ_i be the largest eigenvalue such that $\theta_i \leq -1$. Then,

$$\alpha_2 \leq n \frac{\theta_0 + \theta_i \theta_{i-1}}{(\theta_0 - \theta_i)(\theta_0 - \theta_{i-1})}.$$

Moreover, this is the best possible bound that can be obtained by choosing a polynomial and applying the ratio-type bound.

Ratio-type bound: best polynomial for k=3

(Neuwman, Sajna and Kavi 2023)

The optimal polynomial for $k = 3$

Theorem (Newman, Sajna and K)

Let G be δ-regular graph with n vertices, adjacency matrix A, and distinct eigenvalues $\delta = \theta_0 > \theta_1 > \cdots > \theta_d$, with $d \geq 2$.
Let s be the largest index such that $\theta_s \geq -\frac{\theta_0^2 + \theta_0\theta_d - \triangle}{\theta_0(\theta_d + 1)}$, where $\triangle = \max_{u \in V}\{(A^3)_{uu}\}$, twice the largest number of triangles on any vertex in G.
Let $b = -(\theta_s + \theta_{s+1} + \theta_d)$ and $c = \theta_d\theta_s + \theta_d\theta_{s+1} + \theta_s\theta_{s+1}$.
Then $p(x) = x^3 + bx^2 + cx$ is an optimal polynomial for $k = 3$. The corresponding bound on the 3-independence number of G is

$$\alpha_3 \leq n\frac{\triangle - \theta_d^3 + b(\theta_0 - \theta_d^2) - c\theta_d}{(\theta_0^3 - \theta_d^3) + b(\theta_0^2 - \theta_d^2) + c(\theta_0 - \theta_d)}.$$

Why optimization using MILPs?

(i) The quantum k-independence number is upper bounded by the inertial-type bound (A., Elphick and Wocjan 2022):

$$\alpha_k \leq \alpha_{kq} \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}.$$

For $k > 1$ we can use the MILPs to compute values of the quantum parameter when the bound is tight. For $k = 1$:

$$\alpha \leq \alpha_q \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

(ii) Closed formulas for small k.

(iii) Use the polynomials involved in the MILPs:
inertial-type bound (A., Coutinho, Fiol 2019)
ratio-type bound (Fiol 2020)
to relate both bounds (A. Dalfó, Fiol, Zeijlemaker 2023+)

Open problems

- Complexity of the MILPs? Does increasing k make the problem easier?
- Use the MILPs for other graphs and values of k, and find more closed formulas for graph families.
- Relationship between inertial-type and ratio-type bounds via the obtained polynomials from the MILPs.
- SDP formulation for the inertial-type bound?

$$\alpha \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}$$
$$\alpha_k \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}$$

Thank you for listening!

Further reading:

A. Abiad, G. Coutinho, M.A. Fiol, B.D. Nogueira and S. Zeijlemaker,
Optimization of eigenvalue bounds for the independence and chromatic number of graph powers
Discrete Math. 345(3) (2022)